In the garden of numbers, I will play.

A Thesis

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in the Department of Mathematics of Mysore University

by

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Title Page

Title of the thesis: In the garden of numbers, I will play.

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Guide's name: Prabhakar Vaidya

Date: February 15, 2012

Certificate page

I, Kishor Bhat, certifiy that this thesis is the result of research work done by me under the supervision of Prabhakar Vaidya at the National Institute of Advanced Studies. I am submitting this thesis for possible award of Doctor of Philosophy (Ph.D.) degree in Mathematics of the University of Mysore.

I further certify that this thesis has not been submitted by me for award of any other degree/diploma of this or any other University.

Signature of Doctoral candidate

Signed by me on February 15, 2012

Signature of Guide

Date : February 15, 2012

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DECLARATION IN CONNECTION WITH RESUBMISSION OF THESIS

I hereby declare that the thesis entitled "In the garden of numbers, I play." Which I am currently resubmitting for the degree of Doctor of Philosophy, is the result of research work carried out by me in the National Institute of Advanced Studies, Bangalore under the guidance of Professor Prabhakar Vaidya, Professor. I further declare that the thesis has been revised on the basis of the evaluation report and is resubmitted.

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I certify that Mr. Kishor Bhat has revised his thesis for the degree of Doctor of Philosophy as required by the evaluation report. I certify that all corrections suggested have been incorporated and the thesis is being resubmitted in its revised form. The revision has been carried out under my guidance.

Date:

PRABHAKAR G. VAIDYA
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Abstract

This work is an exploration of certain questions pertaining to the sequence 1, 2, 3, 4, 5, ... It is divided into three parts. The first part stems from a project conducted in 2007, headed by K. Ramachandra (Professor at the National Institute of Advanced Studies). We will give the details of the project here.

Godfrey Harold Hardy remarked, in his lectures on Srinivasa Ramanujan's life that Ramanujan, when he was a child in school, discovered that $e^{i\theta} = cos(\theta) + i\ sin(\theta)$ *(from which the relation* $e^{i\pi} + 1 = 0$ *comes as a consequence). He did this entirely on his own. The exact age at which he made this discovery is not known, but Hardy places his age between 7 and 16. In the book,* The Music of the Primes, *Marcus du Sautoy says that, after this discovery, Ramanujan "found out a few days later that Euler had beaten him to this great discovery by some hundred and fifty years. Humbled and dispirited, Ramanujan hid his calculations in the roof of his house." Ramanujan's original proof and method are unknown. Ramanujan had no access to any material of modern mathematics save one, Sidney Luxton Loney's* Trigonometry*. It is a mystery how Ramanujan was able to make use of any ideas of complex numbers with the nuances necessary to reach Euler's equation. What resulted from this project was an introductory pamphlet on Trigonometry, where we begin the simple equations* $(a+b)^2 = a^2 + 2ab + b^2$ *or* $(a-b)^2 = a^2 - 2ab + b^2$ *(one can start from either), and reach Euler's formula. This is the probable proof of Ramanujan. This has been published in* Mathematics Student*.*

The second part of the thesis is a simplification of many important problems in the analytic theory of numbers. Here, we provide a discussion of many of the current aspects of the theory, such as the Riemann zeta function and the Lindelöf hypothesis. and explain

them in elementary terms. We will also discuss a paper of ours published in the Hardy Ramanujan Journal *and relate it to these problems.*
The third part, which is new, is a generalization of a problem posed by Paul Erdős in 1993 in American Mathematical Monthly. *Consider the equation*

$$n! = a!b!$$

where $n > b > a > 1$. *This equation has an infinite number of solutions. A trivial solution can be constructed as follows: For any arbitrary natural number* a, $n = a!$, *and* $b = a! - 1$. *Nontrivial solutions exist, for example,* $10! = 6!7!$. *Is there a finite number of non-trivial solutions? In a paper by Florian Luca it was proven conditionally using a weaker form of the famous "ABC conjecture," that there is a finite number of nontrivial solutions: but the "ABC conjecture" – being a relative of the Riemann hypothesis – may be a long way off from proving. The question reduces to finding a bound on the distance between* n *and* b. *A trivial solution would be of the form* $n - b = 1$. *Erdős was able to obtain an upper bound of the difference to* $5 \log \log n$. *We improve the absolute constant to* $\frac{1+\epsilon}{\log 2}$ *and generalize this result, with the same constant, to the equation* $n! = a_1!a_2! \dots a_k!$, *for finite* k. *This result has been accepted for publication in the Russian journal* Matematicheskie Zametki. *We proceed with a modification of this theorem where we obtain comparable bound when we substitute the factorial function with the product of terms in a class of arithmetic progressions, that is* $d(2d)(3d)(md)$ *in the place of* $m!$: *The bound we obtain is* $\frac{1+\epsilon}{\log p}$, *where* p *is the greatest prime factor of the common difference in the arithmetic progression.*

Dedicated to

Pál Erdős

and

Srinivasa Ramanujan

for their legacy and their inspiration

Acknowledgements

I try to think of all of the people that I am grateful to, that made this possible. If the list were to be truly comprehensive, I would have to make this section longer then the rest of the work. To limit it down to this section is truly a challenge, and as I miss some people, I hope it is understood that it is not because I do not appreciate you, but for this specific project I can only thank so many.

The first person to come to mind is Professor K. Srinivas from IMSc. It was he who suggested that I look that a paper by Erdős that made the whole work possible. He has been amazingly helpful in every interaction that I have had. It is because of his suggestions at various points of time that I was able to make as much progress as I have, and considering the actual number of times I have met him, it is truly amazing how much of a difference he was able to make for me in this phase of my career. I would also like to thank the other members of my academic "family," specifically A. Sankaranarayanan and N. Sharada from TIFR and Ajai Choudhury, who explained many things to me in my interactions with them, and Michel Waldschmidt from Université Pierre et Marie Curie Paris 6 Institut de Mathématiques de Jussieu who gave me a substantial amount of time to understand certain aspects of Transcendental Number Theory. While these topics are not central to my thesis, it has helped me grow in my academic endeavors. I also would like to thank the members of the Mathematical Modelling Group at NIAS who have been especially encouraging.

I would like to thank all of my school teachers who have built me up to this level. They are usually the unsung heroes. Specifically I will single out Mr. Kesari, Ms. Sneha Titus and Ms. Shapiro who I remember fondly as being especially encouraging in my interest in mathematics. I would like to thank Vidhu Prasad, James Graham-Eagle, and Kostya Rybnikov at University of Massachusetts Lowell, who helped me get my start at research mathematics during my Masters.

I am gratefully, and irreparably indebted to my two PhD advisors, K. Ramachandra and P. G. Vaidya. I have had better than I could have ever hoped for in both of them. They provided the right level of all qualities and encouragement that no student would deserve. It is they who taught me how to think, a gift that only a teacher can give. I have been fortunate to find a single person who could have given me such a gift, let alone two. Professor Ramachandra has been a spectacular example, meeting with me daily and making sure my work progressed in a proper fashion. In many ways this is our thesis, not just mine. Professor Vaidya, and his wonderful family had given me much needed support in all respects.

I would like to thank DST for funding my doctoral program at NIAS. I am very fortunate to have been part of the NIAS community. The students, staff and faculty have provided a stimulating and friendly atmosphere. I have learned a lot of many topics while here. For the sake of brevity I will not go into all of their names one by one, but one name deserves special mention: Ms. J. N. Sandhya has been with me through every step of this thesis making sure that it saw its way to fruition.

As with any project of this nature, it would not have been possible without the support of my family. My parents, Prabhakara and Vijayalaxmi Bhat, and my sister Vrinda have contributed to making me who I am. They nurtured my mathematical interests and made sure I could come to this point in my life. Without their support, I would not have made it here. I also want to thank my companion, Aswathy Raveendran, for her love and support, and for keeping me sane in the second half of my doctoral tenure.

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Keywords

Number Theory, Riemann Zeta function, Diophantine Equations, Erdős, Hickerson, Factorials, Trigonometry, Elementary Problems, Ramanujan, Hardy, Littlewood, Weyl, Hurwitz Zeta function

Chapter 1

The Garden

ಸಂಖ್ಯೆಗಳ ತೋಟದಲ್ಲಿ ನಾನು ಆಡುತ್ತೇನೆ

The theory of numbers is a grand expanse of problems that one can get lost in the various questions one can seek the answer. Over the course of my years as a doctoral candidate, I have been fortunate to have worked in such an area. The title of this thesis was originally intended to be ಸಂಖ್ಯೆಗಳ ತೋಟದಲ್ಲಿ ನಾನು ಆಡುತ್ತೇನೆ (*samkhyagaḷa toṭadalli naanu aaḍuttee, English: In the garden of numbers, I will play*). It is important to note here, that toṭa is a wide expanse where many things grow. The toṭa, which means either garden, grove or plantation, has a special cultural connotation, where many things grow along side each other and are taken from the tree to be appreciated. The toṭa of numbers also is rich with ideas, challenges and applications.

ಸಂಖ್ಯೆಗಳ ಹೂತೋಟದಲ್ಲಿ ನಾನು ಆಡುತ್ತೇನೆ (*samkhyagaḷa huutoṭadalli naanu aaḍuttee, English: In the flower garden of numbers, I will play*) was another considered title for this thesis. As Professor Ramachandra would often tell me, the theorems of mathematics should not be thought of as jewels to be seen in the sky, but rather should be seen as plants that are cultivated in a garden. Mathematical truths must be found growing and carefully cultivated over time. Mathematics is not done however to feed our mouths or to build our houses. Mathematics stands as a intellectual exercise, where our minds stand to a challenge.

Before we begin with section 1, let us take a few minutes to visit our garden. I want to begin by showing what this visit could be like and how this visit can prepare us for the outline of this thesis. This is because the garden is a garden of numbers. Numbers and their properties to most can be quite prosaic. So, let us begin our journey of this garden. Before you are making up your mind whether to spend time in this garden, let me show you what attracted me and many others to this garden. This attraction would be obvious to anyone who enters this garden because soon after you enter the garden, you see a magnificent tree. Gazing at beautiful trees is only half of the fun.

Why is this important? Just seeing is not enough. A tree can be an illusion. With the rivers, the waterfalls and the myriad of Sun's rays glistening into the rising mist, we cannot always trust what we see. We have to find a path to the tree and to touch it. Only then we would know that that the tree is for real.

The explorers have spent lifetimes preparing detailed maps that explain how to reach the tree and touch it and verify it for themselves, provided, of course, they are willing to take the time to train themselves to follow their maps. There were a lot of explorers and there are a lot of these maps. You have to have a large set of them with you before you read a new map. Because, the maps will usually begin by saying, follow such and such's map to reach this particular tree and from there to reach my tree, turn left...

Before we finally begin, I have to tell you about Ramanujan. He was such a child at heart. Just like children often do, he never walked, only danced. He touched so many magnificent trees but at times we do not know what paths he took to reach them. It must be clear to you that it helps to carry a lot of maps as you enter this garden. Our prince had just one or two tiny maps. Other visitors would often lose sight of him, and yet, all of a sudden he would appear atop some mound, touching a beautiful tree. "How did he get there? And with only such old maps!" I have been looking at his maps, gazing at what he could achieve with them. All I have succeeded in doing was to look at *one* pretty tree. We know now that he was not the first one to see it. That credit goes to Euler. But, he was blissfully unaware of Euler and most other explorers and most of the standard maps!

One thing that my advisor and I tried to do is to play this game. We only used the maps that the prince had. Can we reach this tree? We found a path. This is nothing new. Euler had another path a long long time ago. Our hope was that once we found a path that the prince could have taken, using his limed maps, maybe we would someday find his other secret paths. Our thesis begins with a journey to get any insight, any insight at all in the search of the prince's pathway.

We present these three distinct chapters which can be treated as self-contained works or as interrelated parts of a bigger whole. The beginning starts at the fundamentals, and builds up to the third chapter where all of the results are completely new.

1.1 Trigonometry

The first chapter is an attempted reconstruction of the proof by S. Ramanujan on the theorem of Euler. It stems from a project conducted in 2007, headed by K. Ramachandra. The project was inspired by a remark by G. H. Hardy in his lectures on Srinivasa Ramanujan's life [22]. He states in the introduction that Ramanujan, when he was a child in school, discovered that $e^{i\theta} = cos(\theta) + i\ sin(\theta)$. He did this entirely on his own. Ramanujan had no access to any material of modern mathematics save one, Sidney Luxton Loney's *Trigonometry*, so Ramanujan's original proof and method is unknown.

From the biographical account provided by P.V. Seshu Aiyar and R. Ramachandra Rao[20], we have the following observations of Ramanujan's life:

> "While he was in the second form he had, it appears, a great curiosity to know the "highest truth" in Mathematics, and asked some of his friends in the higher classes about it. It seems that some mentioned the Theorem of Pythagoras as the highest truth, and that some others gave the highest place to "Stocks and Shares"... While in the third form, when his teacher was explaining to the class that any quantity divided by itself was equal to unity, he is said to have stood up and asked if zero divided by zero was also equal to unity. It was about this time he mastered the properties of the three progressions... While in the fourth form, he took to the study of Trigonometry. He

> borrowed a copy of the second part of Loney's *Trigonometry* from a student of the B.A. class, who was his neighbour. This student was struck with wonder to learn that this young lad of the fourth form had not only finished reading the book but could do 'every problem in it without any aid whatever; and not infrequently this B.A. student used to go to Ramanujan for the solution of difficult problems. While in the fifth form, he obtained unaided Euler's Theorems for the sine and the cosine and, when he found out later that the theorems had been already proved, he kept the paper containing the results secreted in the roofing of his house."

This indicates that Ramanujan, at this age, was familiar with the Pythagorus' theorem. His knowledge of a proof could have been based on diagram, but that does not explain any of his further interest in trigonometry. He had learnt about arithmetic, geometric and harmonic progression and some basic theorems regarding them. It is at this time, that Ramanujan, mostly likely, had developed at least the precursors to the method that he was to use later in his life. A few years after this, he obtained a copy of Carr's *Synopsis*, and the rest is history.

Given what knowledge he possessed, it remains as an interesting exercise to restrict one's knowledge to the information that was available to him and try to reconstruct his probable proof. Ramanujan's proof came as inspired by beginning of Loney's *Trigonometry*, so one has to take Loney's *Trigonometry* as a point of reference. Assuming that Ramanujan could develop basic ideas of these series (eg: logarithmic, exponential, and trigonometric) from Loney's book, and had a method of manipulating series that was highly intuitive, we developed the presentation of the first chapter as a proposed reconstruction of his original book. Though it is not intended to be original research in the sense of producing new results, the presentation shows how many results can come as applications of ideas seen in the first few sections. This could shed some light on the early methods of Ramanujan. The presentation, though covering many basic topics is unconventional, starting with basic diagrams and leading up to a theory of series, which could be refined as a student matures. While speculative, the presentation is consistent with the biographical details of Ramanujan's life and could very easily have been his method.

As a prerequisite, Loney assumes that the reader knows the theory of similar triangles and Pythagorean theorem. While a conventional curriculum usually starts with the former and then proves the later, we give a presentation that takes the opposite direction, in effect showing that the two statements are equivalent, and grounding the theory in geometric intuition rather then algebraic formalism.

As the motivation of Ramanujan's subsequent work was in number theory, not geometry, rest of the chapter shows a development from basic geometric intuitions and proceeds to a theory of series.

The result was an introductory pamphlet to trigonometry which is published as the first chapter of this thesis. While not every step in this pamphlet can be obviously justified as being Ramanujan's, we do see some early aspects in this work. First is the connection of geometry to gain insight into number theory. This can be seen in "Squaring the circle" (published in 1913) and in "Modular equations and approximations to π" (published in 1914). Second is the development of a theory of series, which was a speciality of his, which eventually led him to do work in Analysis. Third is a connection with the order presented in Loney's *Trigonometry*, which eventually leads to the theorem in question. As Ramanujan was pursuing the theorem independently, he would have had to been slightly unconventional, but not fully deviating from the text. Otherwise the rediscovery would not have come as such surprise. Fourth is the anticipation of certain ideas that would appear later in his life. We ignore certain aspects of his work that were clearly derived from later sources. For example, his paper on "Some properties of Bernoulli's numbers," relies on material that he picked up from Edwards' *Differential Calculus*, a book he only saw after this period. We consider that he would have anticipated certain ideas of calculus (those that could be derived using basic understandings of limits and algebra), but do not proceed far enough so that an understanding of calculus of complex variables is obvious.

To compliment the material presented, as well as link it to further sections of the thesis, sections have been placed in the appendix making the presentation complete and self-contained. In principle, the entire thesis should be accessible to an undergraduate student of mathematics, and quite possibly to a motivated school student, as Ramanujan was.

1.2 On Prime Numbers

The second part is a simplification of many important problems in analytic theory of numbers. Here, we provide a discussion of many of the current aspects of the theory, such as the Riemann zeta function, Lindelöf hypothesis, etc. and explain them in elementary terms. To keep in the rhythm of the first chapter, many basic theorems of analysis are mentioned explicitly, and their proofs are pushed to the appendix. The appendix of this thesis acts in a way to complete the thesis to keep it self-contained. We have ensured that even the most basic theorems are proven so as to make the piece self contained. Theorems mentioned throughout the thesis are proven at length with the exception of those in section 3.5, where appropriate referencing has been made. By bringing up these topics, we motivate material for the final section section of this chapter. The theory of the Riemann-zeta function has implications on prime number theory. In this chapter we discuss a theorem of Ingham which implies that for all N large enough the inequality:

$$N^3 < p \leq (N+1)^3$$

is solvable for some prime p. (It should be noted that the corresponding theorem for squares is an open question, even if we assume the Riemann Hypothesis.) Ingham's result was that:

$$\pi(x+h) - \pi(x) \sim \frac{x^c}{\log x}$$

where $\pi(x)$ is the number of prime numbers less then x and c is any constant greater then $\frac{5}{8}$. Ingham's theorem requires an explicit formula for $\sum_{p \leq x} \log p$. This explicit formula uses the functional equation, but an alternative approach is due to by the introduction of Hooley-Huxley contour.

The main work in the present section is to sketch a proof of this without using the functional equation of $\zeta(s)$.

1.3 Factorials

The third part is a generalization of a problem posed by Paul Erdős in 1993 in *American Mathematical Monthly*. Consider the equation

$$n! = a!b!$$

where $n > b > a > 1$. Solutions can be constructed as follows: For arbitrary natural numbers, take such numbers as a, define n as the factorial of a and define $b!$ as one less of n. By inspection one can see that this is a solution to the above equation. These solutions are considered to be trivial [18]. Till now only one nontrivial solutions has been found (ie: $10! = 6!7!$). It has been shown that for $n < 410$ that this is the only example[18]. It is an open question as to whether this is the only counter example. One way to approach this is to find a bound on the distance between n and b. A trivial solution would be of the form $n - b = 1$. Erdős was able to obtain an upper bound of the difference to $5 \log \log n$. We improve the absolute constant to $\frac{1+\epsilon}{\log 2}$ and generalize this result, with the same constant, to a more general equation of the form $n! = a_1!a_2! \dots a_k!$, for finite k. We do this by developing a function that counts the number of factors of 2 in each term. From that we will develop upper and lower bounds for the function and conclude that $n - b < \frac{(\log_2 b) \log((n-b) \log n)}{\log b - \log((n-b) \log n)}$.

We proceed with a modification of this theorem where we obtain comparable bound when we substitute the factorial function with the product of terms in a class of arithmetic progressions, that is $d(2d)(3d)(md)$ in the place of $m!$. To attack this problem, we develop an analogous proof using a function that counts the factors of d. The bound we obtain is $\frac{1+\epsilon}{\log p}$, where p is the greatest prime factor of the common difference in the arithmetic progression.

Chapter 2

Trigonometry

2.1 Area

We will assume or define (in this case it is the same) that the area of a square with side equal to 1 has an area equal to 1, and if two rectangles have the same dimensions (that is height and width), then the area covered by one rectangle is equal to the area covered by the other. We prove three lemmas and we will conclude with a lemma that a rectangle with sides of length a and b will have area ab.

LEMMA 1: Given two rectangles, $R1$ and $R2$, such that the heights of $R1$ and $R2$ are the same, and the width of $R1$ is a non-negative integral multiple, say ρ times, of $R2$, then the area of $R1$ is ρ times the area of $R2$.

LEMMA 2: Given two rectangles, $R1$ and $R2$, such that the height of $R1$ and $R2$ are the same, and the width of $R1$ is a non-negative rational multiple, say ρ times, of $R2$, then the area of $R1$ is ρ times the area of $R2$.

LEMMA 3: Given two rectangles, $R1$ and $R2$, such that the height of $R1$ and $R2$ are the same, and the width of $R1$ is any non-negative real multiple, say ρ times, of $R2$, then the area of $R1$ is ρ times the area of $R2$.

LEMMA 4: Given two rectangles, $R1$ and $R2$, such that the height of $R1$ is a non-negative real multiple, say ρ_1 times, of $R2$, and the width of $R1$ is a non-negative real multiple, say ρ_2 times, of $R2$, then the area of $R1$ is $\rho_1\rho_2$ times the area of $R2$.

Lemma 4 follows from Lemmas 1, 2 and 3, by appropriate interchanging of height and width. Lemma 3 follows from Lemma 2, as for any given non-negative real number, say ρ, there is a sequence of rational numbers ρ_j that will approximate it, and consequently the area of $R1$ will be approximated by a sequence of rectangles $R1_j$ such that the area of $R1_j$ is ρ_j times the area of $R2$. Since ρ_j approximates to ρ, then the area of $R1_j = \rho_j R2$ approximates $R2$.

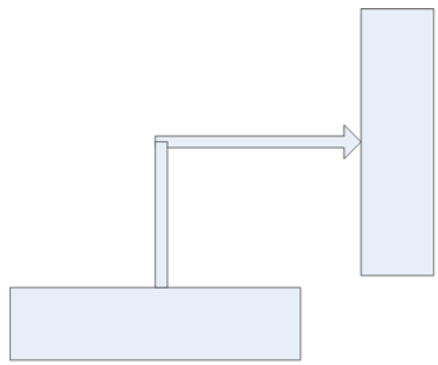

Figure 2.1: Lemma 4

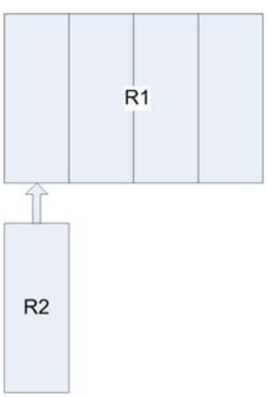

Figure 2.2: Lemma 3

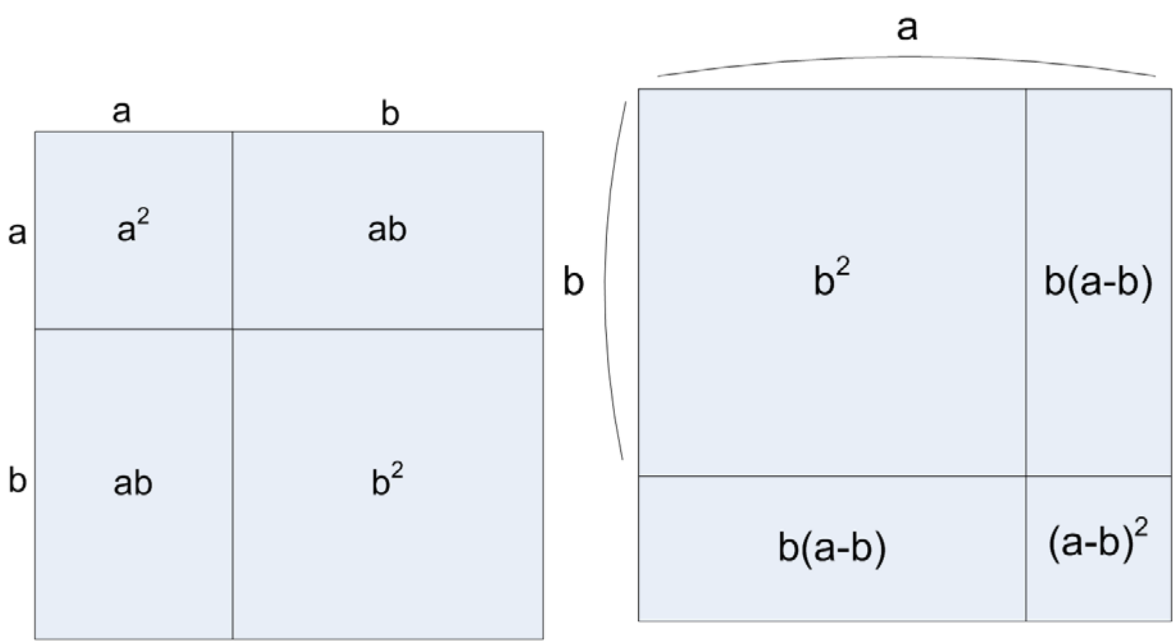

Figure 2.4: The algebraic equations follow from these figures.

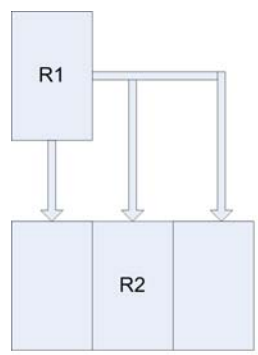

Figure 2.3: Lemma 2

Lemma 2 follows from Lemma 1. If the rational number ρ is a unit fraction, then by interchanging $R1$ and $R2$ we can apply Lemma 1. If ρ is an integral multiple of a unit fraction, then we can apply Lemma 1 on the result we just got and get Lemma 2. To prove Lemma 1, we simply arrange ρ copies of $R2$ along the base. The area of the copies will overlap $R1$ exactly, and the area of the copies, and hence the area of $R2$ will be $\rho R1$.

Figure 1.4 can be taken as proofs that $(a+b)^2 = a^2+2ab+b^2$ and $(a-b)^2 = a^2-2ab+b^2$.

2.2 Pythagoras' Theorem

Pythagoras' Theorem states that given a triangle $\Delta QBC(\hat{Q} =$ a right angle), the area of the square subtended from side BC is equal to the sum of the areas of the squares

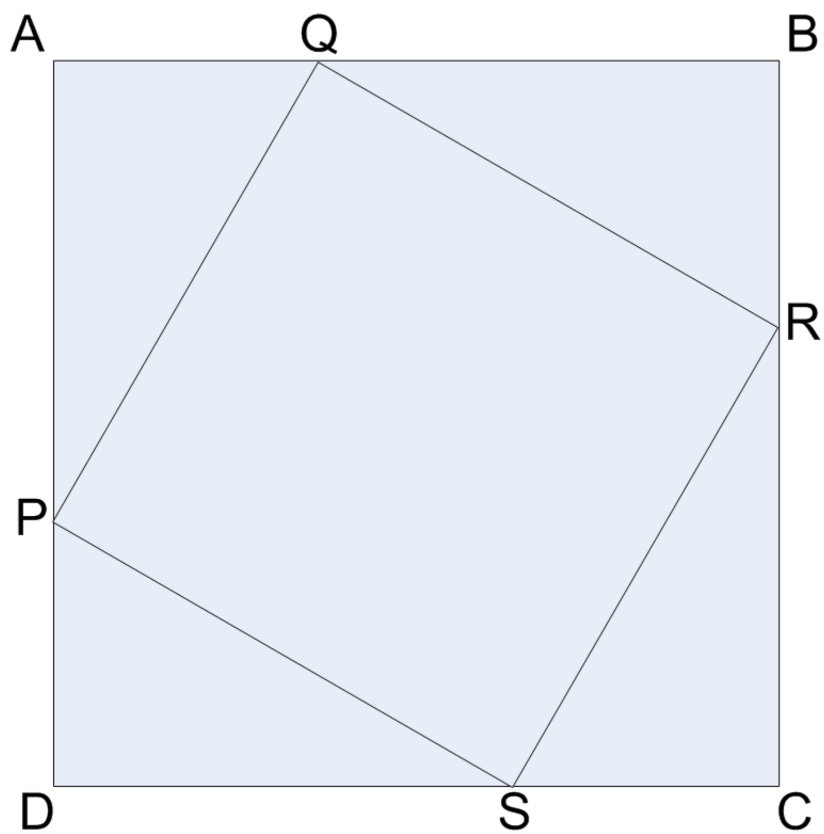

Figure 2.5: First Proof

subtended from the other two sides. In this section, we give two proofs of this Theorem, one based on $(a+b)^2 = a^2 + 2ab + b^2$ and the other on $(a-b)^2 = a^2 - 2ab + b^2$ (both these identities are valid for any two numbers a, b).

First Proof (using $(a+b)^2 = a^2 + 2ab + b^2$). Consider the triangle Δ SRC, where $\hat{C}$ is a right angle. Draw the square PQRS. Draw Δ QBR in such a way that $CR = QB$ and $SC = RB$. Similarly draw the triangles Δ PAQ and ΔPDS. All the four triangles are equal in area (since three sides of one equal three sides of the other). Each of these triangles is equal in area to that of Δ SCR, which is plainly equal to $\frac{1}{2}$ CR.SC. The four triangles together make up $4\frac{1}{2}CR \times SC = 2CR \times SC$. Also they are equal to $(CD)^2 - (SR)^2$. Thus (since $CD = DS + SC = (CR + SC)$),

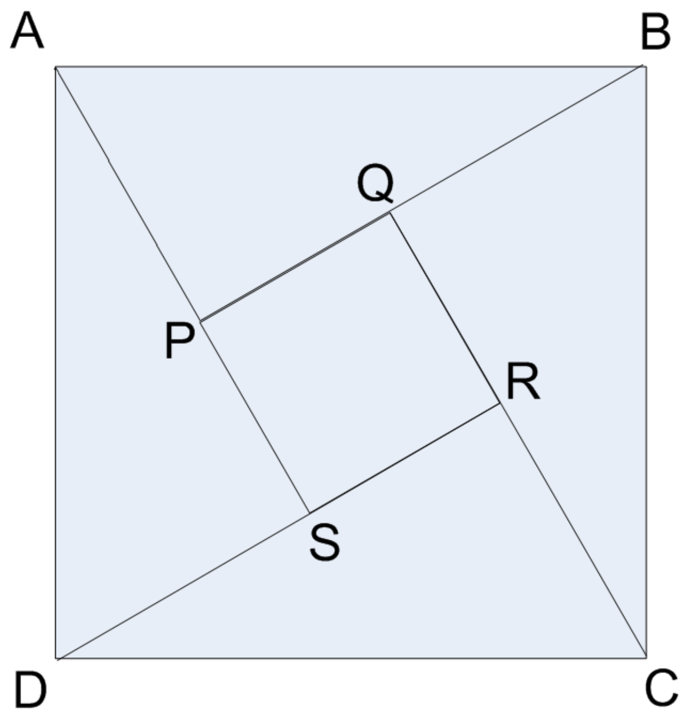

Figure 2.6: Second Proof

$$2CR \times SC = (CR + SC)^2 - (SR)^2.$$

Hence

$$\begin{aligned} (SR)^2 &= (CR)^2 + 2CR \times SC + (SC)^2 - 2CR \times SC \\ &= (CR)^2 + (SC)^2. \end{aligned}$$

Second Proof(using $(a - b)^2 = a^2 - 2ab + b^2$). Consider the triangle ΔABP right angled at $\hat{P}$. Construct a triangle $\Delta QBC(\hat{Q} =$ a right angle) such that $BQ = AP$ and $CQ = BP$. Plainly $PQ = BP - AP$ (since ΔABP is congruent to ΔQBC). Certainly $AB = BC$. Do the same thing for triangle ΔRDC and ΔADS. The four small triangles make up an area of $4.\frac{1}{2}AP \times BP = 2AP \times BP$. Hence

$$\begin{aligned}(AB)^2 &= (BP - AP)^2 + 2AP \times BP \\ &= (BP)^2 + (AP)^2,\end{aligned}$$

using $(a - b)^2 = a^2 - 2ab + b^2$.

REMARK We have used the fact that the area of a right angled triangle is equal to half the product of the sides containing the right angle. We will amplify this remark again in section 1.3.

2.3 Similar Triangles

In this section we prove that Pythagoras' theorem implies that if two triangles are similar then their corresponding sides are proportional. By similar we mean that the three angles of one are equal to the three angles of the other. In order to prove this we can not do any thing better than reproducing from the reference [38]. We deduce the general case from the following theorem as a corollary.

Theorem 1 *If two right angled triangles are similar then their corresponding sides are proportional.*

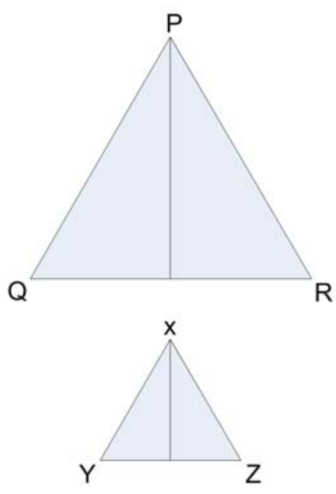

Figure 2.7: Two Similar Triangles

To prove the corollary, divide each of the similar triangles ΔPQR and ΔXYZ into right angled triangles as shown in figure 1.7 and apply theorem 1 twice

While, there is a well-known method of proving Pythagoras' Theorem using theorem 1, but it is less well known that deduction can be made in the opposite direction namely: we can prove theorem 1 using Pythagoras' theorem. The following proof may be new.

Let ΔABC and ΔDEF be similar right angled triangles. If they are congruent then there is nothing to prove, so assume without loss of generality that the length of DE is less than the length of AB. Superimpose ΔDEF on ΔABC in such a way that the point D coincides with the point A and DE, DF fall respectively on AB, AC as in figure 1.8. Draw EG perpendicular to BC.

Write $AB = c, BC = a, CA = b, AE = \lambda c, AF = \mu b$, where $0 < \lambda < 1$ and μ is positive. It suffices to show that $\lambda = \mu$ since E lies between A and B, F cannot lie outside AC, for otherwise the parallel lines EF and BC would intersect. Hence $0 < \mu < 1$.

Also write $BG = x, GC = EF = y$. Applying Pythagoras' theorem we have

$$x^2 = (c - \lambda c)^2 - (b - \mu b)^2$$
$$y^2 = \lambda^2 c^2 - \mu^2 b^2$$

$$\text{and } a^2 = c^2 - b^2. \tag{2.1}$$

Now $x = a - y$, so $x^2 = a^2 + y^2 - 2ay$. Hence $2ay = a^2 + y^2 - x^2$. Substituting from (1.1) we obtain

$$2ay = 2a\lambda c^2 - 2\mu b^2$$

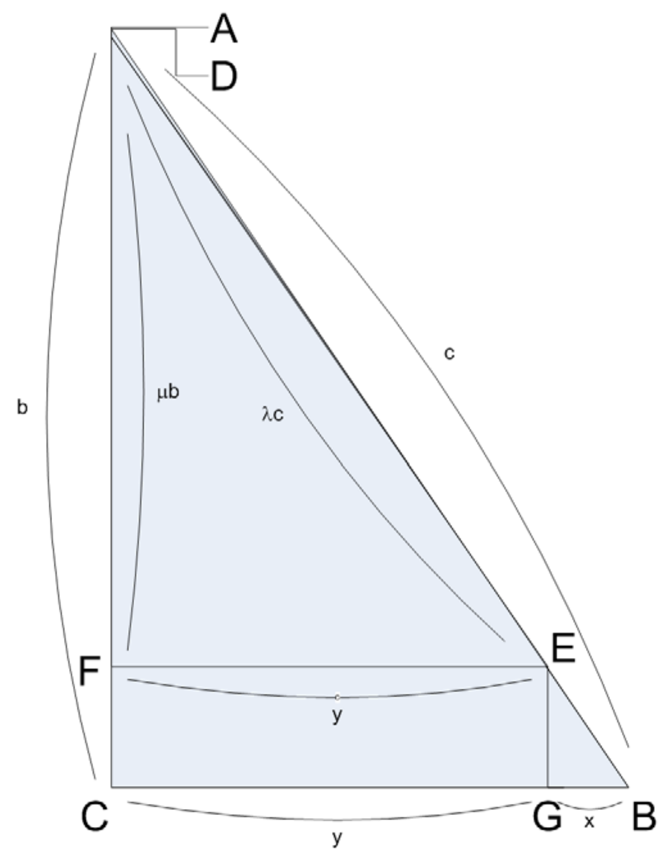

Figure 2.8: Similar Triangle Theorem Proof

Hence

$$a^2 y^2 = (\lambda c^2 - \mu b^2)^2.$$

Substituting again for a^2 and y^2 from (1.1) we obtain after simplification $(\lambda - \mu)^2 b^2 c^2 = 0$ and hence $\lambda = \mu$ as required.

REMARK The proof of the theorem on similar triangles reproduced here is a simplified version (due to the referee) of the paper by K.Ramachandra [38].

2.4 Addition formula for sine and cosine functions

From the results of section 1.3 it is clear that the ratios

$$\sin\theta = \frac{\textit{opposite side}}{\textit{hypotenuse}} \textit{ and } \cos\theta = \frac{\textit{adjacent side}}{\textit{hypotenuse}}$$

are absolute quantities which do not depend on the right angled triangle. The same is true of

$$\tan\theta = \frac{\textit{opposite side}}{\textit{adjacent side}}, \quad cot\ \theta = (\tan\ \theta)^{-1}$$

and other ratios cosec $\theta = (\sin\theta)^{-1}, sec\ \theta = (\cos\theta)^{-1}$.

It is very surprising that these ratios have an addition theorem.

ADDITION THEOREM *We have*

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

and

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

where α, β and $\alpha+\beta$ are positive angles (all less than a right angle).

COROLLARY 1. *Let $i = \sqrt{-1}$. Then*

$$\begin{aligned} & \cos(\alpha+\beta) + i\sin(\alpha+\beta) \\ = \;& \cos\beta(\cos\alpha + i\sin\alpha) + \sin\beta(i\cos\alpha - \sin\alpha) \\ = \;& \cos\beta(\cos\alpha + i\sin\alpha) + i\ \ \sin\beta(\cos\alpha + i\sin\alpha) \\ = \;& (\cos\alpha + i\sin\alpha)\ (\cos\beta + i\sin\beta) \end{aligned}$$

COROLLARY 2. *Let θ be any angle. Then for all positive integers n*

$$\cos\theta + i\sin\theta = (\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})^n$$

(Note: RH makes sense whatever θ (positive angle) provided n is large). Also if we take $cos^2\theta + \sin^2\theta = 1$ to be valid for negative θ and interpret $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ we will have $\cos^2\theta + \sin^2\theta = 1 = (\cos\theta + i\sin\theta)(\cos(-\theta) + i\sin(-\theta))$. Hence we can uphold corollary 2 to negative integers n also.

PROOF OF THE MAIN THEOREM (Proof of addition theorem for $\sin\theta$ due to I.M.Gelfand [17])

We begin with

LEMMA 1. *Area of any triangle is equal to half the product of any two sides multiplied by the sine of the included angle.*

PROOF Suppose the triangle is ΔABC. Draw AD perpendicular to BC. Area of ΔABC is equal to the sum of the areas of ΔABD and ΔADC.

$$= \frac{1}{2}BD \times AD + \frac{1}{2}DC \times AD$$

$$= \frac{1}{2}AD(BD + DC) = \frac{1}{2}AD \times BC.$$

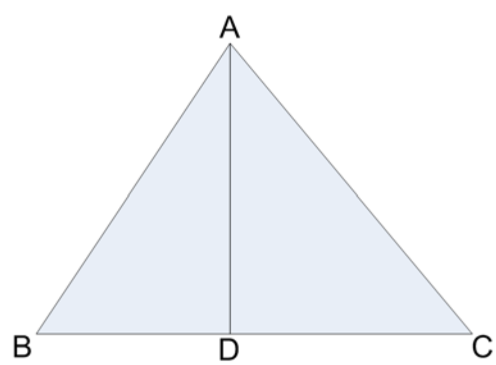

Figure 2.9: Area of a triangle

But $AD = AB\times$ sine of the angle $A\hat{B}D$. Hence area of $\Delta ABC = \frac{1}{2}AB \times BC\times$ sine of the angle $A\hat{B}D$ and this proves the lemma.

REMARK. We have used the fact that the area of a right-angled triangle (say ΔACD) is $\frac{1}{2}DC \times AD$ which is half of the area of the rectangle ABCD (see the figure 1.10)

Consider the figure 1.11, where angle $B\hat{A}D = \alpha$ and $D\hat{A}C = \beta$, and AD is perpendicular to BC. Select B such that angle $B\hat{A}D = \alpha$ and C such that $D\hat{A}C = \beta$. Plain by $B\hat{A}D = \alpha + \beta$. Area of $\Delta BAC = \frac{1}{2}AB \times AC\ sin(\alpha + \beta)$. Also area of $\Delta BAC =$ the sum of the areas ΔBAD and ΔDAC.

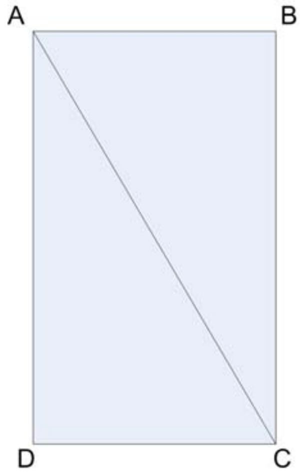

Figure 2.10: Rectangle ABCD

$$= \frac{1}{2} AD \times BD + \frac{1}{2} AD \times DC.$$

Hence $AB \times AC \sin(\alpha + \beta) = AD \times BD + AD \times DC$. But $AD = AC \cos\beta = AB \cos\alpha$ and $BD = AB \sin\alpha$ and also $DC = AC \sin\beta$.

There follows $AB \times AC \sin(\alpha + \beta) = AC \cos\beta \times AB \sin\alpha + AB \cos\alpha \times AC \sin\beta$. Canceling $AB \times AC$ throughout we obtain $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$. Next (using $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ which follows from a consideration of right angled isosceles triangle)

$$\begin{aligned}
\cos\theta &= \sin\left(\frac{\pi}{4} + \frac{\pi}{4} - \theta\right) \\
&= \sin\frac{\pi}{4} \cos\left(\frac{\pi}{4} - \theta\right) + \cos\frac{\pi}{4} \sin\left(\frac{\pi}{4} - \theta\right) \\
&= \sin\frac{\pi}{4} \sin\left(\frac{\pi}{4} + \theta\right) + \cos\frac{\pi}{4} \sin(\frac{\pi}{4} - \theta) \\
&= \sin\frac{\pi}{4} (\sin\frac{\pi}{4} \cos\theta + \cos\frac{\pi}{4} \sin\theta) + \cos\frac{\pi}{4} \sin(\frac{\pi}{4} - \theta) \\
&= \frac{1}{2} \cos\theta + \frac{1}{2} \sin\theta + \frac{1}{\sqrt{2}} \sin(\frac{\pi}{4} - \theta)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{\sqrt{2}} \sin\left(\frac{\pi}{4} - \theta\right) &= \frac{1}{2} \cos\theta - \frac{1}{2} \sin\theta \\
i.e\ \sin\left(\frac{\pi}{4} - \theta\right) &= \frac{1}{\sqrt{2}} (\cos\theta - \sin\theta).
\end{aligned}$$

From this we obtain (by using $\cos\theta = \sin(\frac{\pi}{2} - \theta)$ and $\sin\theta = \cos(\frac{\pi}{2} - \theta)$

$$\begin{aligned}
\cos(\theta+\phi) &= \sin\left(\frac{\pi}{4}-\theta+\frac{\pi}{4}-\phi\right) \\
&= \sin(\frac{\pi}{4}-\theta)\cos(\frac{\pi}{4}-\theta)+\cos(\frac{\pi}{4}+\theta)\sin(\frac{\pi}{4}-\phi) \\
&= \sin(\frac{\pi}{4}-\theta)\sin(\frac{\pi}{4}+\phi)+\sin(\frac{\pi}{4}+\theta)\sin(\frac{\pi}{4}-\phi) \\
&= \frac{1}{\sqrt{2}}(\cos\theta-\sin\theta)\frac{1}{\sqrt{2}}(\cos\phi+\sin\phi)+\frac{1}{\sqrt{2}}(\sin\theta+\cos\theta)\frac{1}{\sqrt{2}}(\cos\phi-\sin\phi) \\
&= \frac{1}{2}(\cos\theta\cos\phi+\cos\theta\sin\phi-\sin\theta\cos\phi-\sin\theta\sin\phi) \\
&+ \frac{1}{2}(\sin\theta\cos\phi-\sin\theta\sin\phi+\cos\theta\cos\phi-\cos\theta\sin\phi) \\
&= \cos\theta\cos\phi-\sin\theta\sin\phi
\end{aligned}$$

which is the addition theorem for $\cos\theta$ namely $\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\phi\sin\phi$. i.e. $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ with a change of notation. This proves our addition theorem completely. It must be remembered that $0\le\alpha\le\frac{\pi}{2}$ and $0\le\beta\le\frac{\pi}{2}$ $0\le\alpha+\beta\le\frac{\pi}{2}$. But we extend it to other (real) values of α and β by using

$$(\cos\theta+i\sin\theta)=\left(\cos\frac{\theta}{n}+i\sin\frac{\theta}{n}\right)^n$$

valid for all integers $n\neq 0$.

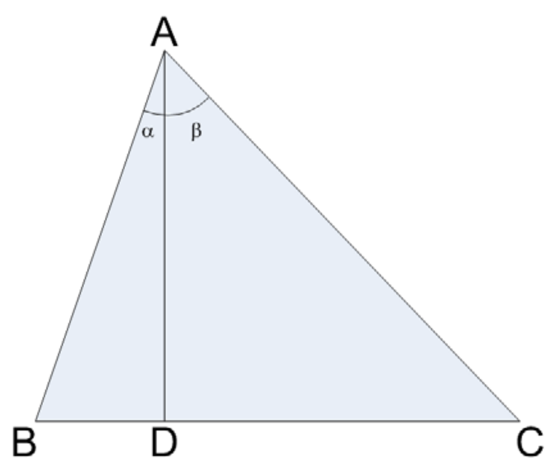

Figure 2.11: $\Delta BAC=\frac{1}{2}.AB.AC.\sin(\alpha+\beta)$.

2.5 Radian Measures and Calculation of Trigonometric Ratios

Although we start with sexagesimal measure such as $30^o, 45^o, 180^o$ and so on, we find it convenient to designate by 2π the circumference of a circle of unit radius. We call the angle around a point 2π radians. We divide the circumference into very small equal parts say k parts – and each part is of length $\frac{2\pi}{k}$. The angle subtended at the center by each part is $\frac{2\pi}{k}$ radians. By choosing k large, we can define (by rule of three) the angle $\theta(0 < \theta \leq 2\pi)$ subtended at the center by arc of length θ will be θ radians. Of course we can calculate π by using

$$tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$$

and putting $x = 1$ and $\tan^{-1} 1 = \frac{\pi}{4}$ (radian measure). These things will be worked out in section 1.7. For the calculation of trigonometric ratios we start with

LEMMA1. *For* $0 < \theta \leq \frac{\pi}{2}$, *we have*

$$\cos\ \theta = 1 + O(\theta^2)$$

NOTE From now on we employ the notation $O(\dots)$ to mean "less than a constant times …". Thus stated in other words the lemma reads

$$|\cos\theta - 1| \leq C\theta^2$$

where $C > 0$ is an absolute constant. From now on, all angles will be in radian measure.

PROOF. $|\cos\theta - 1| = 1 - \cos\theta = \frac{1-\cos^2\theta}{1+\cos\theta} \leq (\sin\theta)^2 \leq \theta^2$ and we prove a more precise result regarding $(\sin\theta)$ in the next lemma.

LEMMA 2. $|\sin\theta - \theta| = O(\theta^3)$,

PROOF. In the figure 1.12, $AB = AC = AF = 1, B\hat{A}C = \theta =$ arc BFC, and AGF is perpendicular to BC. DFE is tangent to arc BFC touching it at F. Also F is the middle point of arc BFC. We recall that the area of any triangle is half the product of any two sides multiplied by the sine of the included angle. Hence by dividing the arc θ into small bits area of sector ABFC=θ. Area of ΔABC is $\frac{1}{2}\ \sin\theta$.

Again $BD = GF = AF - AG = 1 - \cos\frac{\theta}{2}, BC = 2\sin\frac{\theta}{2}$. Hence

$$0 \ < \ \frac{1}{2}\theta - \frac{1}{2}\sin\theta$$

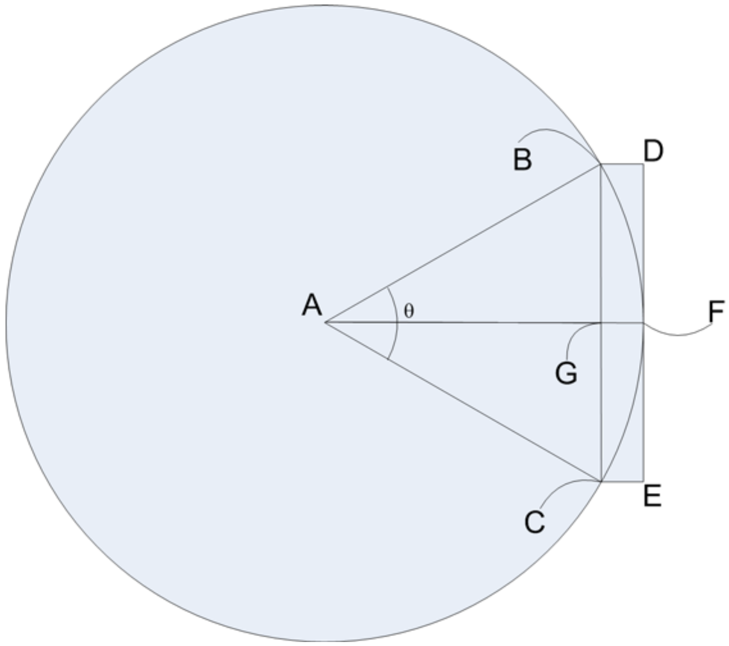

Figure 2.12: $|\sin\theta - \theta| = O(\theta^3)$

$$\begin{aligned}
&\leq (2\sin\frac{\theta}{2})(1-\cos\frac{\theta}{2}) \\
&\leq (2\sin\frac{\theta}{2})(1-\cos^2\frac{\theta}{2})(1+\cos\frac{\theta}{2})^{-1} \\
&\leq \theta(\sin\frac{\theta}{2})^2 \\
&\leq \frac{1}{4}\theta^3
\end{aligned}$$

Thus $0 < \theta - \sin\theta \leq \frac{1}{2}\theta^3$. This proves the lemma. Our next step is

THEOREM *If θ is in radian measure then*

$$(\cos\theta + i\sin\theta) = (1+\frac{i\theta}{n})^n + O(\frac{1}{n^2}).$$

PROOF. We begin with the identity

$$\begin{aligned}
& A^n - B^n \\
= \ & (A-B)(A^{n-1} + A^{n-2}B + A^{n-3}B^2 + \ldots + B^{n-1}) \\
\leq \ & |A-B|(nJ^n)
\end{aligned}$$

where $J = \max(|A|, |B|)$. We observe

$$\begin{aligned}
\cos\theta + i\sin\theta &= \left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)^n \text{ and also} \\
\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)^n - \left(1 + \frac{i\theta}{n}\right)^n &= A^n - B^n, \left(\text{where A} = \cos\frac{\theta}{\text{n}} + \text{i}\sin = \frac{\theta}{\text{n}} \text{ and B} = 1 + \frac{\text{i}\theta}{\text{n}}\right), \\
|A^n - B^n| \leq |A - B| nJ^n \text{ where J} &= \max(|\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}|, |1 + \frac{i\theta}{n}|) \leq 1 + \frac{C_o}{n}
\end{aligned}$$

(where C_o is positive and constant, and so $nJ^n \leq n(1 + \frac{C_o}{n})^n \leq Dn$, where D is a positive constant. The inequality $(1 + \frac{C_o}{n})^n \leq D$ for all large n will be proved in the next section).

$$\begin{aligned}
|A - B| &= |\cos\frac{\theta}{n} - 1 + i(\sin\frac{\theta}{n} - \frac{\theta}{n})| \\
&\leq |\cos\frac{\theta}{n} - 1| + |\sin\frac{\theta}{n} - \frac{\theta}{n}| \\
&= O(\frac{1}{n^2} + \frac{1}{n^3}) \\
&= O(\frac{1}{n^2}), \text{by using lemmas 1 and 2.}
\end{aligned}$$

Thus

$$\begin{aligned}
&\cos\theta + isin\theta - (1 + \frac{i\theta}{n})^n \\
= &(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})^n - (1 + \frac{i\theta}{n})^n \\
= &O(\frac{D}{n} .n .\frac{1}{n^2}) = O(\frac{1}{n^2})
\end{aligned}$$

and this proves the theorem.

LEMMA *We have the inequality*

$$\sum_{n=0}^{\infty} \frac{C^n}{n!} \leq \sum_{0 \leq n \leq 100C^3} \frac{C^n}{n!} + 4$$

where C is any positive constant.

PROOF Put $K = 100C^3$ (without loss of generality we can assume that C is a positive integer). We can ignore $\sum_{0\le n\le K} \frac{C^n}{n!}$. Now

$$\begin{aligned}
\sum_{n\ge K+1} \frac{C^n}{n!} &\le \sum_{n\ge K+1} \frac{C.C\ldots, C \textit{ to } n \text{ terms}}{100C^3.100C^3\ldots \textit{ to } n \text{ terms}} \\
&\le \sum_{n\ge K+1} \frac{C^2.C^2.\ldots C^2 \; to[\frac{n}{2}]+1 \text{ terms}}{100C^3.100C^3\ldots to[\frac{n}{2}]+1 \text{ terms}} \\
&\le \sum_{n\ge K+1} \frac{1}{100C.100C\ldots[\frac{n}{2}]+1 \text{ terms}} \\
&\le \sum_{n=0}^{\infty} 2.2^{-n} = 4. \text{ since 100C} \ge 4 \text{ and}
\end{aligned}$$

$$4.4.\ldots \text{to } [\frac{\mathrm{n}}{2}]+1 \text{ terms } \ge 2.2^{-\mathrm{n}}.$$

Letting $n \to \infty$ we have

COROLLARY. $\lim_{n\to\infty} \left(1+\frac{i\theta}{n}\right)^n$ *exists and is equal to* $\cos\theta + i\sin\theta, \theta$ *being in radian measure.*

REMARK. The notion of a limit(and consequently the concept of convergence of a sequence (such as infinite series, infinite products and infinite continued fractions)) depends on the notion of a distance of a real or a complex number from the origin. This necessitates the notion of the real line or the complex plane as the case may be. The distance of real number a being $|a|$ and that of a complex number $x + iy$ being the positive square root $\sqrt{x^2+y^2}$.

Also

$$x + iy \;=\; \sqrt{x^2+y^2}\; e^{itan^{-1}\frac{y}{x}+2\pi ik}$$

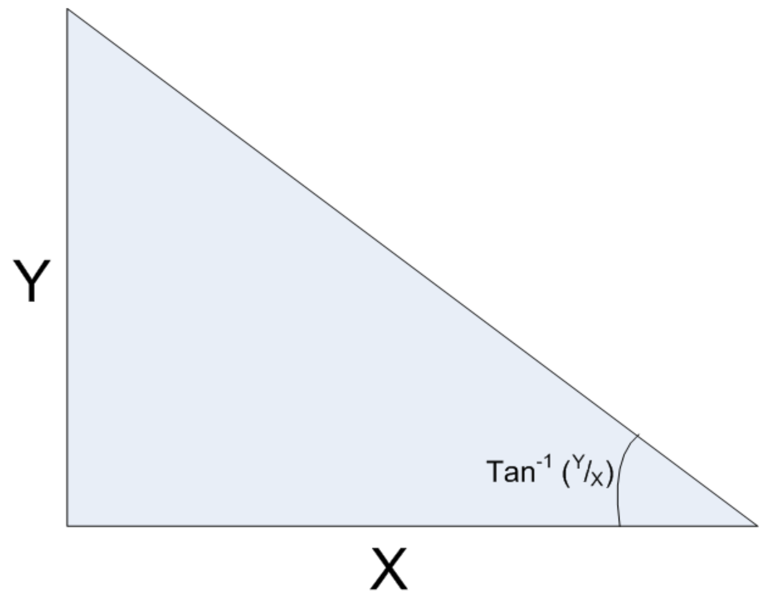

Figure 2.13: The bottom angle of the above triangle is $\tan^{-1}\frac{y}{x}$

$$= \sqrt{x^2+y^2}\left\{\frac{x}{\sqrt{x^2+y^2}} + \frac{iy}{\sqrt{x^2+y^2}}\right\}.$$

Hence

$$\log(x+iy) = \log\sqrt{x^2+y^2} + \text{i}\tan^{-1}\frac{y}{x} + 2\pi k \quad \text{k} = 0, \pm 1, \pm 2, \ldots$$

These are in some ways a natural way of introducing the distance function (also called Archimedean valuation). There are other ways (called non-Archimedean valuations). For example

$$|2^n|_2 = 2^{-n} \; and \; |6^n|_2 = 2^{-n}.$$

You will surely raise your eyebrows if I say that the distance of 2^n from the origin is 2^{-n} and so 2^n tends to zero as $n \to \infty$. But you need not. There is a rich subject called "p-adic analysis". Every rational number has a p-adic distance called "p-adic valuation" associated with every prime p. For example if $(p, 6) = 1$ then $|6|_p = 1$. Also we can do differentiation, integration theory of analytic functions and so on. In this theory, the radius of converge of $\sum\limits_{n=1}^{\infty} \frac{z^n}{n^1}$ is not infinity but $p^{-\delta}, \delta = \frac{1}{p-1}$, !!!

This is certainly not perverted intelligence. The extra-ordinary results of A.Baker have been extended to p-adic valuations [45]. These give very rich dividends to ordinary Diophantine equations and more general diophantine problems. The best Indian expert on p-adic analysis is Professor T.N. Shorey. He has worked on p-adic transcendence and application of Baker's work to diophantine questions. From now on we use only the ordinary distances of real and complex numbers from the origin.

In the next section we will express $\lim_{n\to\infty}(1+\frac{i\theta}{n})^n$ as an infinite power series in powers of $i\theta$. Separating real and imaginary parts we obtain series for the $\sin\theta$ and $\cos\theta$ in powers of θ.

2.6 Exponential Series

The object of this section is to prove the following theorem.

THEOREM 1 *For any fixed complex number $z = x + iy$ we have*

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = 1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots.$$

PROOF. For proving this theorem we need the expansion $(1+\frac{z}{n})^n = 1+\frac{n}{1!}(\frac{z}{n})+\frac{n(n-1)}{2!}(\frac{z}{n})^2+\frac{n(n-1)(n-2)}{3!}(\frac{z}{n})^3+\dots$ to $n+1$ terms, where n is a positive integer (this could have been proved by Ramanujan).

MOTIVATION We can start with

$$(1-x)^{-1} = 1+x+x^2+\dots.$$

differentiate both sides with respect to x n times. We get successively (for LHS)

$$1!(1-x)^{-2}, 2!(1-x)^{-3}, 3!(1-x)^{-4}, \dots \textit{ to } n \textit{ terms}$$

RHS will be

$$1.x^0+2.x^1+3x^2+4x^3+\dots$$
$$2.1.x^0+2.3x^1+3.4x^2+\dots$$

We can guess what happens at the r^{th} stage and we get an expansion for

$$(1-x)^{-r}.$$

Here we can replace r by $-n$ and x by $-x$ and thus we can get a formula for

$$(1+x)^n.$$

From this we can get the expansion for $(1+\frac{z}{n})^n$, (which is the well known Binomial theorem for a positive integral index) stated earlier.

We need a lemma

LEMMA

$$\begin{aligned} &|\frac{n(n-1)}{n^2}-1| \le \frac{2^2}{n},\ |\frac{n(n-1)(n-2)}{n^3}-1| \le \frac{2^3}{n}, \\ &|\frac{n(n-1)(n-2)(n-3)}{n^4}-1| \ \le \frac{2^4}{n}, \ldots \end{aligned}$$

PROOF We have for $0 < r < n$

$$|\frac{n(n-1)(n-2)\ldots(n-r)}{n^{r+1}}-1| \le |\frac{(n-1)(n-2)\ldots(n-r)}{n^r}-1|$$

(Note that the first term is positive and less than 1 and so the quantity in question is)

$$\begin{aligned} |\frac{n(n-1)(n-2)\ldots(n-r)}{n^r}-1| &\le |(\frac{n-r}{n})^r-1| \\ &= n^{-r}(n^r-(n-r)^r) \\ &= n^{-r}(n-(n-r))(\sum_{j=0}^{r-1} n^j(n-r)^{r-1-j}) \\ &\le n^{-r}.r.r\ n^{r-1} \\ &= \frac{r^2}{n} \\ &= \frac{r^2}{2}.2.\frac{1}{n} \\ &\le \frac{2^{r+1}}{n} \end{aligned}$$

$$\text{since } 2^{\mathrm{r}} \geq 1 + \mathrm{r} + \frac{\mathrm{r}(\mathrm{r}-1)}{2} = 1 + \frac{\mathrm{r}^2 + \mathrm{r}}{2} \geq \frac{\mathrm{r}^2}{2}.$$

This proves the lemma.

Now

$$\begin{aligned}
|(1+\frac{z}{n})^n - (1+\frac{z}{1!}+\frac{z^2}{2!}+\ldots)| &\leq |(1+\frac{|z|}{n})^n - \sum_{m=0}^{n}\frac{|z|^m}{m!}| + \sum_{m>n}\frac{|z|^m}{m!} \\
&\leq (\frac{|z|^2}{2!}\frac{2^2}{n} + \frac{|z|^3}{3!}\frac{2^3}{n} + \frac{|z|^4}{4!}\frac{2^4}{n} + \ldots + \frac{|z|^n}{n!}\frac{2^n}{n}) \\
&\quad + \sum_{m>n}\frac{|z|^m}{m!} \\
&= P + Q \; say.
\end{aligned}$$

It is clear that

$$\begin{aligned}
Q &\leq \frac{|z|^n}{n!}\sum_{m=n+1}^{\infty}\frac{|z|^{m-n}n!}{m!} \\
&= \frac{|z|^n}{n!}\sum_{r=1}^{\infty}\frac{|z|^r n!}{(n-r)!} \\
&\leq \frac{|z|^n}{n!}\sum_{r=1}^{\infty}\frac{|z|^r}{r!}
\end{aligned}$$

since for $r \geq 1$ we have $\frac{n!}{(n+r)!} \leq \frac{1}{n!}$ $(\frac{(n+r)!r!}{n!}$ being a binomial coefficient in $(1+x)^{n+r}$ and hence an integer). Hence if $|z| \leq C$(we have for some $C > 2$)

$$Q \leq \frac{C^n}{n!}\sum_{r=1}^{\infty}\frac{C^r}{r!} \leq \frac{C^n}{n!} \textit{ times a constant.}$$

Hence as $n \to \infty$, $Q \to 0$ as is evident from

$$\frac{C^n}{n!} \leq \frac{C^n}{[\frac{n}{2}]\ldots[n]} \leq \left(\frac{C^2}{\frac{n}{2}-1}\right)^{\frac{n}{2}}$$

valid for $n \geq 100C^2$.

Again

$$\begin{aligned} P &\leq \frac{1}{n}\left\{\frac{|2z|^2}{2!}+\frac{|2z|^3}{3!}+\dots\right\} \\ &\leq \frac{1}{n}\sum_{m=1}^{\infty}\frac{|2z|^m}{m!} \\ &\leq \frac{1}{n}\sum_{m=1}^{\infty}\frac{|2C|^m}{m!} \text{ for } |z| \leq C \end{aligned}$$

It is not hard to prove that the last infinite sum is bounded by a constant depending on C (see the remark before section 1.6).

Hence $P \to 0, Q \to 0$ *as* $n \to \infty$, and this proves the theorem.

REMARK. We do not go in detail to the theory of infinite series. For our purposes an infinite series of complex terms $\sum_{n=1}^{\infty} a_n$ represents a complex number if it is "convergent". For our purposes a series is said to be convergent if the tail portion

$$\sum_{n=m}^{\infty} a_n$$

tends to zero as $m \to \infty$. In our case

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is convergent. (In fact it can be differentiated term by term any number of times). Call this $e(z)$.

$$e(z_1)e(z_2) = \lim_{n\to\infty} \left\{(1+\frac{z_1}{n})^n(1+\frac{z_2}{n})^n\right\}$$

and the limit can be seem to be

$$\lim_{n\to\infty} \left\{1+\frac{z_1+z_2}{n}\right\}^n = e(z_1+z_2)$$

and so

$$(e(z))^n = e(nz), (e(1))^z = \lim_{n\to\infty} \{(1+\frac{1}{n})^{zn}\}$$

$$= \lim_{n\to\infty} \{(1+\frac{z}{n})\}^n. \quad \text{Hence } e(z) = e(1)^z,\ e(1)$$

is usually denoted by e.

Thus we have the following theorem.

THEOREM 2 *We have* $\cos\theta + i\sin\theta = \lim_{n\to\infty}(1+\frac{i\theta}{n})^n = e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \ldots$ *whereby*

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} - + \ldots$$

and

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - + \ldots$$

REMARK. If a and z are complex numbers we have to interpret a^z as $\exp\{z\log a\}$. Since $\log a_1 = \log a_2$ will happen with $a_1 = a_2 e^{2\pi ik}$ for any integer k, we have to specify the logarithm. In case a is a positive real number $\log a$ is in uniquely defined (in practice) as the unique real solution of $e^b = a$. But even in this case $\log a$ is in general any of the numbers $b + 2ki\pi (k = 0, \pm 1, \pm 2, \ldots,)$. If $a(\neq 0)$ is complex we write $\frac{a}{|a|} = e^{i\theta}$ (note the LHS has absolute value 1.) If $A + iB$ is of absolute value 1 so is its square $A^2 + B^2 = 1$, where A and B are real.

2.7 Logarithmic Series

We have seen that

$$\lim_{n\to\infty}(1+\frac{z}{n})^n = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = e^z.$$

Let $1-x$ ($|x| < 1$) be the series on the RHS. It is possible to invert and find an expression for z in terms of x. Put $(1+\frac{z}{n})^n = 1 - x$. There follows

$$\lim_{n\to\infty} n(((1-x)^{\frac{1}{n}} - 1) = z.$$

If we use binomial theorem for the (non-integral) index $\frac{1}{n}$, we have

$$\begin{aligned} z &= \lim_{n\to\infty} n\left(-x.\frac{1}{n} + \frac{(-x)^2}{2!}.\frac{1}{n}\,(\frac{1}{n}-1) + \frac{-x^3}{3!}.\frac{1}{n}(\frac{1}{n}-1)(\frac{1}{n}-2) + \ldots\right) \\ &= \lim_{n\to\infty}(-x - \frac{x^2}{2} - \frac{x^3}{3} + \ldots + Q_n) \end{aligned}$$

where

$$Q_n = \{-(1-\frac{1}{1.n})\frac{x^2}{2} + \frac{x^2}{2} - (1-\frac{1}{1.n})(1-\frac{1}{2.n})\frac{x^3}{3} + \frac{x^3}{3}$$
$$-(1-\frac{1}{1.n})(1-\frac{1}{2.n})(1-\frac{1}{3.n})\frac{x^4}{4} + \frac{x^4}{4} + \ldots\}$$

Hence

$$\begin{aligned} |Q_n| &\leq \frac{|x|^2}{2}(1-(1-\frac{1}{1.n})) + \frac{|x|^3}{3}(1-(1-\frac{1}{1.n})(1-\frac{1}{2.n})) + \ldots \\ &= \sum_{r=2}^{\infty}\left\{\frac{|x|^r}{r}((1-(1-\frac{1}{1.n})(1-\frac{1}{2.n})\ldots(1-\frac{1}{(r-1)n}))\right\} \\ &\leq \sum_{r=2}^{\infty}\frac{|x|^r}{r}\left|(1+\frac{1}{1.n})(1+\frac{1}{2.n})\ldots(1+\frac{1}{(r-1)n}) - 1\right| \\ &\quad \left(\text{since } (1+\frac{1}{1.\text{n}})(1+\frac{1}{2.\text{n}})\ldots\ (1+\frac{1}{(\text{r}-1)\text{n}})\right. \\ &+ \left.(1-\frac{1}{1.n})(1-\frac{1}{2.n})\ldots\ (1-\frac{1}{(r-1)n}) \geq 2\right) \\ &\leq \sum_{r=2}^{\infty}\frac{x^r}{r}\left\{\frac{1}{n}\sum_{j\leq r}\frac{1}{j} + \frac{1}{n^2}(\sum_{j\leq r}\frac{1}{j})^2 + \ldots \text{to r terms}\right\} \\ &\leq \sum_{r=2}^{\infty}\frac{x^r}{r}\left\{\frac{1}{n}(\log r + O(1)) + \frac{1}{n^2}(\log r + O(1))^2 + \ldots \text{to r terms}\right\} \\ &= \sum_1 + \sum_2 \end{aligned}$$

where $\sum_1$ is over those r with $n > (10\log r + O(1))^2$ i.e. $(\log r + O(1)) \leq 5\sqrt{n}$ and $\sum_2$ those with the remaining r namely $r > N, N = e^{(\frac{n}{10}+O(1))}$.

$$\begin{aligned}
\sum_1 &\leq \sum_{r=2}^{\infty} \frac{|x|^r}{r}\left\{\frac{1}{n}5\sqrt{n} + \frac{1}{n^2}(5\sqrt{n})^2 + \text{ to r terms}\right\} \\
&\leq \sum_{r=2}^{\infty} \frac{|x|^r}{r}\left\{\frac{5}{\sqrt{n}} + (\frac{5}{\sqrt{n}})^2 + (\frac{5}{\sqrt{n}})^3 + \ \dots\ to\ \infty\right\} \\
&= \sum_{r=2}^{\infty} \frac{|x|^2}{r}\left\{\frac{\frac{5}{\sqrt{n}}}{1-\frac{5}{\sqrt{n}}}\right\} = \sum_{r=2}^{\infty} \frac{|x|^r}{r}\left\{\frac{5}{\sqrt{n}-5}\right\} \\
&\leq \sum_{r=2}^{\infty} \frac{|x|^r}{r}.\frac{10}{\sqrt{n}}\ if\ n\ \sqrt{n}-5 \geq \frac{\sqrt{n}}{2}\ i.e.\ if\ n \geq 100. \\
&\leq \frac{|x|^2}{1-|x|}.\frac{10}{\sqrt{n}}.
\end{aligned}$$

Next

$$\begin{aligned}
\sum_2 &\leq \frac{1}{N}\sum_{r\geq N}|x|^r(1+\frac{1}{n})^{r-1}\text{see the third step of inequality for}|\mathrm{Q_n}| \\
&\leq \frac{1}{N}\sum_{r\geq 0}|x|^r(1+\frac{1}{n})^{r-1} \\
&\leq \frac{n}{N}(1+\frac{1}{n})^{-1}\left\{1-|x|(1+\frac{1}{n})\right\}^{-1} \\
&= \frac{2n}{N}\left\{1-|x|(1+\frac{1}{n})\right\}^{-1},\ \text{provided } |x|(1+\frac{1}{\mathrm{n}}) < 1, \\
&\leq N^{-\frac{1}{2}}\left\{1-|x|(1+\frac{1}{n})\right\}^{-1}\ \text{ since } \mathrm{N} = \mathrm{e}^{\frac{\mathrm{n}}{10}+\mathrm{O}(1)}.
\end{aligned}$$

Thus we have proved the following

THEOREM. *Let* $|x| < 1$ *and*

$$n((1-x)^{\frac{1}{n}}-1) + \sum_{k=1}^{\infty}\frac{x^k}{k} = E(x).$$

Then

$$|E(x)| \leq \frac{10|x|^2}{1-|x|}\ n^{\frac{1}{2}} + N^{-\frac{1}{2}}\left\{1 - |x|(1+\frac{1}{n})\right\}^{-1}$$

where $n \geq 100$ *and* $N = e^{\frac{n}{10}+O(1)}$.

COROLLARY. Letting $n \to \infty$ we have for $|x| < 1$, the equality $\lim_{n\to\infty} n((1-x)^{\frac{1}{n}} - 1) = -\sum_{k=1}^{\infty} \frac{x^k}{k} = \log(1-x)$.

An alternative approach is to assume an expansion of the following type (in this approach binomial theorem for a non-integral positive index is not necessary. We need "multinomial theorem" for a positive integral index). We start with

$$(1+\frac{z}{n})^n = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \frac{1}{1-x}\ (|x| < 1)$$

$$e^z = \frac{1}{1-x} = 1 + x + x^2 + \ldots |x| < 1$$

Assume that $z = a_0 + a_1x + a_2x^2 + \ldots$. There follows

$$\frac{1}{1-x} = 1 + \frac{(a_0 + a_1x + a_2x^2 + \ldots)}{1!} + \frac{(\ldots)^2}{2!} + \frac{(\ldots)^3}{3} + \ldots.$$

Equating constant terms we get $1 = e^{a_0}$ and so $a_0 = 0$. Equating coefficients of x we have $1 = a_1$. Equating coefficients of x^2 we have $1 = \frac{a_2}{1!} + \frac{a_1^2}{2!} i.e.\ a_2 = \frac{1}{2}$ and so on. By induction we may complete the proof that

$$\log(\frac{1}{1-x}) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots$$

But we will not pursue this proof and do error estimations.

2.8 Trigonometric Functions and their Inverses

We start with the diagram of triangle ΔABC (See figure 1.14) where angle B (denoted by $\hat{B}$) $= \frac{\pi}{2}$. It is clear that $\sin^{-1} x = \hat{C}, \cos^{-1} x + \sin^{-1} x = \hat{A} + \hat{C} = \frac{\pi}{2}, \tan^{-1} \frac{x}{\sqrt{1-x^2}} = \hat{C},\ cot^{-1} \frac{x}{\sqrt{1-x^2}} = \hat{A}$ and so $\tan^{-1} \frac{x}{\sqrt{1-x^2}} + \cot^{-1} \frac{x}{\sqrt{1-x^2}} = \hat{C} + \hat{A} = \frac{\pi}{2}$ and so on. These require the condition $0 < x < 1$ (but relaxable by "Analytic continuation", a term which we do not explain).

From

$$\cos\theta + i\sin\theta = e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots$$

Follows (on equating real and imaginary points),

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - + \dots \text{and}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - + \dots$$

The usual (nice) series for $tan^{-1}x$ can be obtained as follows.

We have

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

Put $z = x + iy$ and we get

$$-\log(1-x-iy) = \sum_{n=1}^{\infty} \frac{(x+iy)^n}{n}.$$

Specializing this to $x = 0$ we obtain

$$-\log(1-iy) = \sum_{n=1}^{\infty} \frac{(iy)^n}{n}.$$

Here LHS $= -\log|1-iy| - \text{i}\tan^{-1}(-y) = -\log|1-iy| + \text{i}\tan^{-1}y$ Thus

$$-\log(1-iy) = \sum_{n=1}^{\infty} \frac{(iy)^n}{n} = -\log|1-iy| + i\ tan^{-1}y$$

Equating imaginary points in the last two formulae we get

$$tan^{-1}y = y - \frac{y^3}{3} + \frac{y^5}{5} - + \dots, \text{where} |y| < 1.$$

But $tan^{-1}1 = \frac{\pi}{4}$ and we recover

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \ldots.$$

by justifying the limit operation $y \to 1$.

So far we have not employed calculus. But we now use calculus.

$$\frac{d}{dy} tan^{-1}y = \frac{1}{1+y^2} \text{ and so } \tan^{-1}y = \int_0^y \frac{\mathrm{d}}{\mathrm{d}y}\ (\tan^{-1}y)dy \text{ and so for } |y| < 1$$

$$= \int_0^y (1 - y^2 + y^4 - \ldots)dy = y - \frac{y^3}{3} + \frac{y^5}{5} - + \ldots$$

Again $\frac{d}{dy}\ \sin^{-1}y = \frac{1}{\sqrt{1-y^2}}$ and so $\sin^{-1}y = \int_0^y\ \frac{dy}{\sqrt{1-y^2}}$

$$= \int_0^y (1 + (-\frac{1}{2})\frac{(-y^2)}{1!} + (-\frac{1}{2})(-\frac{1}{2} - 1)\frac{(-y^2)^2}{2!} + (-\frac{1}{2})(-\frac{1}{2} - 1)(-\frac{1}{2} - 2)\frac{(-y^2)^3}{3} + \ldots)\ dy$$

$$= y + \frac{1}{2}\frac{y^3}{3} + \frac{1.3}{2.4}\ \frac{y^2}{5} + \frac{1.3.5}{2.4.6}\ \frac{y^7}{7} + \ldots.$$

Also $cos^{-1}y = \frac{\pi}{2} - sin^{-1}y = \frac{\pi}{2} - (y + \frac{1}{2}\frac{y^3}{3} + \frac{1.3}{2.4}\ \frac{y^5}{5} + \ldots).$

We end this chapter by proposing a new method for a nice expansion of $(F(x))^k$ where k is any positive integer constant. We limit ourselves to $(tan^{-1}x)^2$ and $(sin^{-1}x)^2$. [Before proceeding further we note (p.203 of part II of the excellent books by S.L.Loney (parts I and II)) the following result.

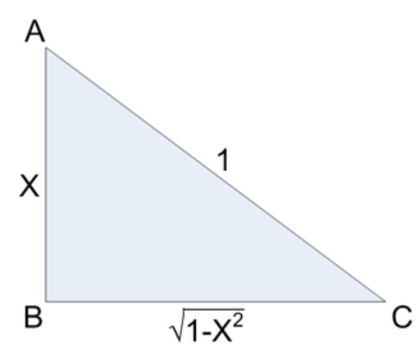

Figure 2.14: Triangle ABC

$$\frac{1}{6}(sin^{-1}x)^3 = \frac{1}{2}\frac{x^3}{3} + (\frac{1}{1^3} + \frac{1}{3^3})\ \frac{1.3}{2.4}\ \frac{x^5}{5} + (\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3})\ \frac{1.3.5}{2.4.6}\ \frac{x^7}{7} + \ldots$$

We hope that we can obtain this by our method.

We now proceed with our method.

Let

$$(tan^{-1}x)^2 = a_0 + a_1x + a_2x^2 + \dots. \tag{2.2}$$

Differentiating both sides with respect to x and multiplying both sides by $1 + x^2$ we get

$$\begin{aligned} 2tan^{-1}x &= (1+x^2)(a_1 + 2a_2x + 3a_3x^2 + \dots) \\ &= 2(x - \frac{x^3}{3} + \frac{x^5}{5} - + \dots) \end{aligned}$$

Equating coefficients of like powers of x, we can obtain $a_0, a_1, a_2, \dots$. Certainly $a_0 = 0$. But in this special case we can get $a_0, a_1, a_2, \dots$ by direct squaring in

$$\left(x - \frac{x^3}{3} + \frac{x^5}{5} - + \dots\right)^2.$$

We now turn to

$$(sin^{-1}x)^2 = a_0 + a_1x + a_2x^2 + \dots \tag{1}$$

differentiating we get

$$\frac{2sin^{-1}x}{\sqrt{1-x^2}} = a_1 + 2a_2x + 3a_3x^2 + \dots \tag{2}$$

One more differentiation gives

$$\frac{2}{1-x^2} + 2(sin^{-1}x)(-\frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x)) = 2.1.a_2 + 3.2.a_3x + 4.3.a_4x^2 + \dots \tag{3}$$

Multiplying throughout by $1 - x^2$, we obtain

$$2 + \frac{2(sin^{-1}x)}{\sqrt{1-x^2}} = (1-x^2)(2.1.a_2 + 3.2.a_3x + 4.3.a_4x^2 + \dots$$

Here we substitute for LHS (using (2))

$$2 + a_1 + 2a_2x + 3a_3x^2 + \ldots$$

$$= (1 - x^2)(2.1.a_2 + 3.2.a_3x + 4.3.a_4x^2 + \ldots). \tag{4}$$

In (4) we equate coefficients of like powers of x and we get $a_1, a_2, a_3, \ldots$ (trivially $a_0 = 0$ and $a_1 = 0$).

We leave it as an exercise for the reader to complete the expansion of $(sin^{-1}x)^2$.

ONE FINAL REMARK: JONATHAN M BORWEIN and MARC CHAMBERLAND have proved the following surprising result.

THEOREM: (1) for $|x| \leq 2$ and $N = 1, 2, 3, \ldots$ we have

$$\frac{1}{(2N)!}\left(\sin^{-1}\frac{x}{2}\right)^{2N} = \sum_{k=1}^{\infty}\frac{H_N(k)}{\begin{pmatrix}2k\\k\end{pmatrix}k^2}x^{2k}$$

where $H_1(k) = \frac{1}{4}$ and

$$H_{N+1}(k) = \frac{1}{4}\sum_{n_1=1}^{k-1}\frac{1}{(2n_1)^2}\sum_{n_2=1}^{n_1-1}\frac{1}{(2n_2)^2}\cdots\sum_{n_N=1}^{n_{N-1}-1}\frac{1}{(2n_N)^2}.$$

(2) For $|x| \leq 2$ and $N = 0, 1, 2, \ldots$ we have

$$\frac{1}{(2N+1)!}(\sin^{-1}\frac{x}{2})^{2N+1} = \sum_{k=0}^{\infty}\frac{G_N(k)\begin{pmatrix}2k\\k\end{pmatrix}}{2(2k+1)4^{2k}}x^{2k+1}$$

where $G_0(k) = 1$ and

$$G_N(k) = \sum_{n_1=0}^{k-1}\frac{1}{(2n_1+1)^2}\sum_{n_2=0}^{n_1-1}\frac{1}{(2n_2+1)^2}\cdots\sum_{n_N=0}^{n_{N-1}-1}\frac{1}{(2n_N+1)^2}$$

the convention is that the sum is zero if the starting index exceeds the finishing index.

The beautiful result

$$\sum_{k=1}^{\infty} \frac{4^k}{\binom{2k}{k} k^3} = \pi^2 \ \log 2 - \frac{7}{8} \ \zeta(3)$$

is deduced in their paper as a special case.

Chapter 3

On Prime numbers

3.1 Complex variables

Imagine you are a teacher at a school, and you want to keep your students busy. To do so you come up with the arduous task of adding up the numbers from 1 to 100. Today any student with a mobile phone can do the task mechanically and produce the answer within minutes. We all know the story of Karl F. Gauss, who was able to devise a method of adding up numbers in a series of this form by developing a theory of addressing the sum of consecutive integers, but in mathematics one finds that there are still interesting series that need to be added.

Take the following series as an example. For each number, count the number of prime factors. If the number of factors is odd, then mark the number as plus. Else mark the number as minus. Start with the number 1. In this case, there are no prime factors, so the number of prime factors is 0. We mark this number as minus. Now take the next number, 2. Again count the prime factors. We get 1. 3 has one prime factor, so we mark these numbers as plus. 4 has two prime factors (2 and 2). So we mark 4 as minus. Continue this procedure until 100. What is the difference between the pluses and minuses?

In the case of 100, we get the number 6 at the end. What of the end of 1,000? 10,000? Calculating this value would be taxing even with the aid of sophisticated computational techniques. An even more basic topic of inquiry is the order of magnitude of this value in terms of the final integer. That is, if we continue this procedure for large numbers, will we find a point where the value we obtain would be consistently overtaken by a function of the number we end with? What is the smallest such function? It is hypothesized that such a function would be $f(n) = \sqrt{n} \log n$. This hypothesis is called the Riemann Hypothesis.

A trivial upper bound is the function $f(n) = n$. Developing a nontrivial upper bound is a very difficult problem. Establishing that the function is even $f(n) = n^{.999999}$ is out of reach, but establishing this would have deep implications for the theory of prime numbers. The current bound is $f(n) = n^{1 - \frac{1}{(\log n)^{2/5} (\log \log n)^{1/5}}}$ [50] [36] [49] . Progressing from this has proven to be a very difficult project.

Very powerful tools have been developed in the analytic theory of numbers. Using these tools, we can see how to convert this problem into a problem of the location of zeros of the an analytic function. We shall present a proof that the above mentioned statement is equivalent to the following statement:

Statement: If the Riemann-zeta function defined by $\zeta(s) = \sum_{n=0}^{\infty} n^{-s}$ where s is complex, equals 0, then the real part of s is $\frac{1}{2}$ or an even negative integer.

We make use of the following statements. The proofs are provided in the appendix.

Cauchy's Integral Theorem If a function f(z) is analytic and one-valued inside and on a simple closed contour C, then $\int_C f(s) ds = 0$.

Cauchy's Residue Theorem Given a function $F(z)$ that can be approximated by the series $F(z) = \sum\limits_{n=-\infty}^{\infty} a_n z^n$, within a circle of unit radius, the value of $\int\limits_C F(z)dz = 0$ if the function is bounded in C. If the function goes to infinity at a single point, z_0 in C, the value of $F(z) = \sum\limits_{n=-\infty}^{\infty} b_n(z - z_0)^n = \int\limits_C F(z)dz = b_{-1}$.

Perron's formula (Due to CHOWLA & BRIGGS [39]) *Let $y > 0$. Then*

$$\frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty} y^s \frac{ds}{s} = \begin{cases} 0 & if \quad y < 1 \\ \frac{1}{2} & if \quad y = 1 \\ 1 & if \quad y > 1 \end{cases}$$

Note: A function is called *analytic* if it can be approximated by a polynomial. A function is called *one-valued* if its output is always a complex number.

We now proceed to show:

LEMMA 1: <u>Let</u> $F(s) = \sum\limits_{n=1}^{\infty} \frac{a_n}{n^s}$, $x > 10$, $X \leq x \leq 30X$, $X \geq T \geq 10$, $c = 1 + \frac{1}{\log x}$
<u>Then</u>

$$\frac{1}{2i\pi}\int_{c-iT}^{c+iT} F(s)\frac{x^s}{s}ds = \sum_{n \leq x} a_n + O(\frac{X}{T}(\log X)^2)$$

PROOF Left Hand Side is

$$\sum_{n \leq x} {}'a_n + O(\sum_{n \leq x-1} a_n | \frac{(\frac{x}{n})^c}{T \log \frac{x}{n}} |)$$

where $'$ denotes that if x is an integer, then

$$\sum_{n \leq x} {}'a_n = \sum_{n \leq x-1} a_n + \frac{1}{2}a_x$$

otherwise it is self explanatory. Hence, whether x is an integer or not.

$$\sum_{n\leq x}{}' a_n = \sum_{n\leq x} a_n + O(1)$$
$$= \sum_{n\leq x} a_n + O(\tfrac{X}{T})$$

The contributions from $n > 2X$, and $n < \frac{x}{10}$ and $|n-x| \leq 3$ to the Right Hand Side are (on noting that $|\log \frac{x}{n}| \geq \frac{|n-x|}{n+x}$ and $|iT+\sigma| \geq T$)

$$\sum_{n=1}^{\infty} \frac{a_n}{T}(\frac{X}{n})^c = O(\tfrac{X\log X}{T(c-1)})$$
$$= O(\tfrac{X(\log X)^2}{T})$$

The remaining portions contribute

$$O(\tfrac{X}{T}\sum_{\frac{X}{10}\leq n\leq 2X}^{|n-x|>3} \tfrac{a_n}{|n-x|}) = O(\tfrac{X}{T}\log X)$$

and hence the lemma is proved.

REMARK: Sometimes it is convenient to replace the Left Hand Side by

$$\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} y^s \frac{ds}{s}$$

and add on an error to the RHS which in absolute value does not exceed

$$min(\tfrac{y^c}{\pi T|logy|}, \tfrac{y^c}{2\pi} + \tfrac{y^c}{\pi}|tan^{-1}\tfrac{c}{T}|)$$

(which does not exceed $2y^c$.)

We do not have to use the error term, but to be very rigorous we can use the error term. One should note that $T = O(X)$ and $X = O(T)$. From this lemma, we get the following lemma as a consequence.

LEMMA 2: *Let $x > 0$. Then*

$$\frac{1}{2i\pi}\int_{2-i\infty}^{2+i\infty} F(s)\frac{x^s}{s}ds = \sum_{n\leq x}\mu(n) + O(1)$$

where

$$F(s) = \frac{1}{\zeta(s)}$$

3.2 Some functions

We introduce the following functions.

For a given integer $n = \prod^{\omega} p^a$, $\omega = \omega(n)$ is the number of distinct prime factors of n, and $\Omega(n) = \sum a$ represents the total number of primes. Consider the number $12 = 2^2.3^1$. $\omega(12) = 2$ and $\Omega(12) = 3$.

Now, consider the function $\zeta^{-1}(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$. Where:

$$\mu(n) = \begin{cases} (-1)^{\omega(n)} & if\ \Omega(n) = \omega(n)) \\ 0 & otherwise \end{cases}$$

Note: $\mu(n)$ is called the Möbius function. A variation of $\mu(n)$, call the Liouville function $\lambda(n) = (-1)^{\Omega(n)}$ can be substituted in, as one can prove that certain inequalities relating to one series carry over to the other very easily. An explanation and justification is given in the appendix.

3.3 Riemann Hypothesis

On application of Perron's formula, if we can assume that $\zeta^{-1}(s) = O(t^{\epsilon})$, then

$$\sum_{k=1}^{x}\mu(k) = \frac{1}{2i\pi}\int_{c-iT}^{c+iT} |\zeta^{-1}(s)|\frac{x^s}{s}ds$$

$$
\begin{aligned}
&= \frac{1}{2i\pi}\int_{\frac{1}{2}+\epsilon_2-iT}^{\frac{1}{2}+\epsilon_2+iT} |\zeta^{-1}(s)|\frac{x^c x^{it}}{\sqrt{c^2+t^2}}ds \\
&= \frac{1}{2i\pi}\int_{-iT}^{iT} |\zeta^{-1}(s)|\frac{x^{\frac{1}{2}+\epsilon_2} x^{it}}{\sqrt{c^2+t^2}}ds \\
&= O(\sqrt{x}\int_{-iT}^{iT} |\zeta^{-1}(s)|\frac{x^{\epsilon_2}}{ct}ds) \\
&= O(\sqrt{x}\int_{-iT}^{iT} t^{\epsilon_1}\frac{x^{\epsilon_2}}{ct}ds) \\
&= O(\sqrt{x}t^{\epsilon_1}x^{\epsilon_2}) \\
&= O(\sqrt{x}x^{\epsilon})
\end{aligned}
$$

Note: Our assumption that $\zeta^{-1}(s) = O(t^{\epsilon})$ can be justified by the fact that the Lindelöf hypothesis is a consequence of the Riemann hypothesis. A proof is enclosed in the appendix.

3.4 Consequences on Theory of Primes

We know that $p_{n+1} - p_n < p^{\theta+\epsilon}$ with some specific $\theta = \frac{1}{2} + \frac{1}{40}$ [2]. In the case of the Lindelöf hypothesis this implies that $p_{n+1} - p_n < p^{\frac{1}{2}+\epsilon}$, or alternatively, for any given ϵ, the number of exceptions to the statement, "Between $n^{2+\epsilon}$ and $(n+1)^{2+\epsilon}$, there is a prime." is finite. In the final section of this chapter we present the Van der Corput version of the proof $\zeta(\frac{1}{2}+it) = O(t^{\frac{1}{6}+\epsilon})$, and apply the technique to prove that $\zeta(\frac{1}{2}+it,\alpha) = O(t^{\frac{1}{6}+\epsilon})$, where $\zeta(s,\alpha) = \sum_{n=0}^{\infty}(n+\alpha)^{-s}$, with $0 < \alpha \leq 1$ is the Hurwitz Zeta function. The proof is not self-contained, and we refer to Titchmarch's book [49] to fill in any gaps.

3.5 A remark on a statement of Ingham

3.5.1 Introduction:

The three main ingredients in the proof of Ingham's Theorems are:

A) I. M. Vinogradov's deep result:

$$\zeta(s) \neq 0 (s - \sigma + it), \quad \sigma \geq 1 - K_1 (logt)^{\frac{-2}{3}} (loglogt)^{\frac{1}{3}},$$

$t \geq 100$, where $K_1 > 0$ is an absolute constant.

B) Explicit formula [28] for

$$\sum_{p \leq x} \log p$$

C)

$$N(\sigma, T) < T^{(\frac{8}{3})(1-\sigma)} (logT)^{100}$$

where $\frac{1}{2} \leq \sigma \leq 1, T \geq 1000$. The precise power of $logT$ is unimportant. Any constant in place of 100 will do. $N(\sigma, T)$ denotes the number of zeros $\beta + i\gamma$ of $\zeta(s)$ with $\beta \geq \sigma$ and $|\gamma| \leq T$.

1) The toughest part is (A). It follows from a deep result (due to Vinogradov)

$$\zeta(\sigma + it) \leq (t^{(1-\sigma)^{3/2}} logt)^{K_2} \qquad (\tfrac{1}{2} \leq \sigma \leq 1, t \geq 100)$$

where $K_2 > 0$ is an absolute constant and Vinogradov's zero-free result follows from this in a relatively simple way by a method due to Landau [36]. For a proof of Vinogradov's upper bound for $|\zeta(\sigma + it)|$ without using the functional equation see [37].

2) Explicit formula uses the functional equation, but an alternative approach is due to [35] by the introduction of Hooley-Huxley contour.

3) The proof of the zero-density bound, stated above, uses

$$\zeta(\tfrac{1}{2} + it) = O(t^{\frac{1}{6}} logt), t \geq 100,$$

where the $O-$constant is absolute. The main work in the present section is to sketch a proof of this without using the functional equation of $\zeta(s)$.

3.5.2 Some Remarks:

In fact we write

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty}(n+\alpha)^{-s}(0<\alpha\leq 1, s=\alpha+it, \sigma>1),$$

and next if X is any positive integer we have

$$\zeta(s,\alpha) = \alpha^{-s} - \sum_{n=0}^{\infty}(n+\alpha)^{-s} + \sum_{n=0}^{\infty}((n+\alpha)^{-s} - \int_{n}^{n+1}\frac{du}{(u+\alpha)^{s}}) + \int_{X+1}^{\infty}(u+\alpha)^{-s}du,$$

Since the last term is

$$\frac{(X+1+\alpha)^{1-s}}{s-1}$$

and the rest is analytic in $\sigma > 0$, we get the analytic continuation in $\sigma > 0$ of $\zeta(s,\alpha)$. We prove our main theorem which is as follows.

THEOREM. We have

$$\zeta(\tfrac{1}{2}+it,\alpha) - \alpha^{-\frac{1}{2}-it} = O(t^{\frac{1}{6}}logt), (t\geq 10)$$

uniformly in the real parameter α. (Note that $\zeta(s,1)=\zeta(s)$).

Proof of the theorem We use Van-der Corput's theorems (Theorems 5.9 and 5.11 of [49]) and after the proof of the theorem we make some comments about the Weyl-Hardy-Littlewood method of proof of our theorem.

Theorem 1 (Theorem 5.9 of [49]) *If f(x) is real and twice continuously differentiable and*

$$0 < \lambda_2 \leq f''(x) \leq h\lambda_2 (or 0 < \lambda_2 \leq -f''(x) \leq h\lambda_2)$$

throughout the interval (a,b) and $b \geq a+1$, then

$$\sum_{a<n\leq b} e^{2\pi i f(n)} = O(h(b-a)\lambda_2^{\frac{1}{2}}) + O(\lambda_2^{-\frac{1}{2}}).$$

Theorem 2 (Theorem 5.11 of [49]) *If f(x) is real and thrice continuously differentiable and*

$$0 < \lambda_3 \leq f'''(x) \leq h\lambda_3 (or 0 < \lambda_3 \leq -f'''(x) \leq h\lambda_3)$$

throughout the interval (a,b) and $b \geq a+1$, *then*

$$\sum_{a<n\leq b} e^{2\pi i f(n)} = O(h^{1/2}(b-a)\lambda_3^{\frac{1}{6}}) + O((b-a)^{\frac{1}{2}}\lambda_3^{-\frac{1}{3}}).$$

We now apply these to

$$E = \sum_{a\leq n\leq b(\leq 2a)} (n+\alpha)^{-it} \text{ with } a \geq 10.$$

Here $f(x) = -\frac{t}{2\pi} log(x+\alpha)$. We have

$$f'(x) = -\frac{t}{2\pi(x+\alpha)}$$
$$f''(x) = \frac{t}{2\pi(x+\alpha)^2}$$
and
$$f'''(x) = -\frac{2t}{2\pi(x+\alpha)^3}$$

Thus

$$C_1 \leq f''(x)a^2t^{-1} \leq C_2$$
$$and\ C_3 \leq f'''(x)a^3t^{-1} \leq C_4$$

where $C_1, C_2, C_3, andC_4$ are absolute positive constants. Thus we have

$$\sum_{a<n\leq b(\leq 2a)} (n+\alpha)^{-it} = O(t^{\frac{1}{2}}) + O(at^{-\frac{1}{2}})$$
$$\sum_{a<n\leq b(\leq 2a)} (n+\alpha)^{-it} = O(t^{\frac{1}{6}}a^{\frac{1}{2}}) + O(t^{-\frac{1}{6}}a).$$

Hence by partial summation we have

$$\sum_{a<n\leq b(\leq 2a)} (n+\alpha)^{\frac{1}{2}-it} = O((\tfrac{t}{a})^{\frac{1}{2}}) + O((\tfrac{a}{t})^{\frac{1}{2}})$$

and

$$t(\sum_{a<n\leq b(\leq 2a)} (n+\alpha)^{-\frac{3}{2}-it}) = O((\tfrac{t}{a})^{\frac{3}{2}}) + O((\tfrac{t}{a})^{\frac{1}{2}}).$$

Also we need

$$\sum_{a<n\leq b(\leq 2a)}(n+\alpha)^{-\frac{1}{2}-it}=O(t^{\frac{1}{6}})+O(t^{-\frac{1}{6}}a^{\frac{1}{2}})$$

which follows from our estimate of $\sum\limits_{a<n\leq b(\leq 2a)}(n+\alpha)^{-it}$. From our estimate of $\sum\limits_{a<n\leq b(\leq 2a)}(n+\alpha)^{-\frac{1}{2}-it}$ it follows

$$\sum_{1\leq n\leq t^{\frac{2}{3}}}(n+\alpha)^{-\frac{1}{2}-it}=O(t^{\frac{1}{6}}logt).$$

From our estimates of $\sum\limits_{a<n\leq b(\leq 2a)}(n+\alpha)^{\frac{1}{2}-it}$ it follows

$$\sum_{t^{\frac{2}{3}}\leq n\leq t^{\frac{4}{3}}}(n+\alpha)^{-\frac{1}{2}-it}=O(t^{\frac{1}{6}}logt).$$

Thus

$$\sum_{1\leq n\leq t^{\frac{4}{3}}}(n+\alpha)^{-\frac{1}{2}-it}=O(t^{\frac{1}{6}}logt).$$

We now fix $X=[t^{\frac{4}{3}}]$. $\frac{(X+1+\alpha)^{1-s}}{s-1}$ contributes $O(t^{-\frac{1}{3}})$. We note that (with $s=\frac{1}{2}+it$)

$$\begin{aligned}&\sum_{n>X}((n+\alpha)^{-s}-\int_{n}^{n+1}\frac{du}{(u+\alpha)^{s}})\\ &=\sum_{n>X}\int_{n}^{n+1}((n+\alpha)^{-s}-(u+\alpha)^{-s})du\\ &=\sum_{n>X}s\int_{0}^{1}(\int_{0}^{u}(n+\upsilon+\alpha)^{-s-1}d\upsilon)du\end{aligned}$$

and so its absolute value is

$$O(\sum_{a>t^{\frac{4}{3}}}((\tfrac{t}{a})^{\frac{3}{2}}+(\tfrac{t}{a})^{\frac{1}{2}}))=O(t^{-\frac{1}{6}}).$$

This proves our main theorem.

Remark 1 Let X be an arbitrary positive integer $\geq 20(|t|+20)(K+1)$. Then by iteration of the method by which we continue $\zeta(s,\alpha)$ in $\sigma>0$ (incidentily the method is due to E. Landau (Handbuch der primzahlen) [30] we can get the analytic continuation in $|\sigma|\leq(K+1)$ (K being arbitrary constant) and also the expression

$$\zeta(s,\alpha)-\alpha^{-s}+\sum_{n\leq X}(n+\alpha)^{-s}+\frac{X^{1-s}}{s-1}+O(X^{-\sigma})$$

where $s=\sigma+it$ (σ arbitrary). (O constant depends on K). For this see [36]

Remark 2 A remark on Weyl-Hardy-Littlewood method is necessary here. The proof of Theorem 5.5 of [49] goes to prove $\sum_{1\leq n\leq t^{\frac{2}{3}}}(n+\alpha)^{-\frac{1}{2}-it} = O(t^{\frac{1}{6}}L)$ except for trivial complications arising from presence of the real parameter α. This uses the integer parameter k to be 2. However if we use the case $k = 1$ simple computations show that

$$\sum_{t^{\frac{2}{3}}\leq n\leq Ct}(n+\alpha)^{-\frac{1}{2}-it} = O(t^{\frac{1}{6}}L)$$

whatever the constant $C \geq 10$ be. Here L is some fixed power of $logt$. These considerations prove the main theorem in view of Remark 1 above. We stress once again that functional equations for $\zeta(s)$ or $\zeta(s,\alpha)$ are not necessary in the proof of Ingham's theorems. L can be any fixed power of log t and this is enough to prove Ingham's asymptotic formula mentioned earlier.

Chapter 4

Factorials

4.1 A problem of Erdős

In a paper published in 1993 in American Mathematical Monthly, Erdős proved that if $n! = a!b!$ with $n > b > a > 1$ then $n - b < 5 \log\log n$ for large enough n, where $\log x$ denotes the natural logarithm of x. We generalize the theorem from two factorials to many. The trivial solutions to this equation are when $n = b + 1 = a!$. So far the only known non-trivial solution to this is $10! = 6!7!$. It is an open problem as to whether the number of non-trivial solutions is finite or not.

4.2 Generalization

The corresponding conjecture can be generalized to more then 2 factorials. That is to say that if $n! = (\prod_{j=1}^{k} a_j!)b!$ with $1 < a_1 \leq a_2 \leq ... \leq a_k \leq b < n$, a trivial solution is $n = b + 1 = \prod_{j=1}^{k} a_j!$. The only known trivial solution to this, barring the one mentioned, is $16! = 14!5!2!$. A good treatment of this problem has been made by Luca. [32]

THEOREM: For arbitrary $\epsilon > 0$, there is n_ϵ depending on ϵ such that for all $n > n_\epsilon$, we have $n! = (\prod_{j=1}^{k} a_j!)b!$ with $1 < a_1 \leq a_2 \leq ... \leq a_k \leq b < n$ then $n - b < \frac{1+\epsilon}{\log 2}(\log\log n)$.

Proof outline: We will first develop a function that counts the number of factors of 2 in each term. From that we will develop upper and lower bounds for the function and conclude that

$$n - b \quad < \quad \frac{(\log_2 b)\log((n-b)\log n)}{\log b - \log((n-b)\log n)}.$$

From this, we will use the inequalities that we derived to obtain the desired result. (Note: We will use $\log_2 n$ to mean $\frac{\log n}{\log 2}$.)

Proof:

First we define a function $\alpha(n) = \sum_{j=1}^{\infty} \lfloor 2^{-j} n \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

A reader should note three things about the function $\alpha(n)$:

$$\begin{aligned}
1 : \alpha(n) \quad & < \quad n \\
2 : \alpha(n) \quad & \text{shows} \quad \text{how many factors of 2 are in the term n!.} \\
3 : \alpha(n) \quad & = \quad \sum_{j=1}^{\infty} \lfloor 2^{-j} n \rfloor \\
& = \quad \sum_{j=1}^{\lfloor \log_2 n \rfloor} \lfloor 2^{-j} n \rfloor \\
& > \quad \sum_{j=1}^{\lfloor \log_2 n \rfloor} (2^{-j} n - 1) \\
& \geq \quad n - \frac{\log n}{\log 2} - 1
\end{aligned}$$

By Statement 2, $n! = (\prod_{j=1}^{k} a_j!)b!$ implies that $\alpha(n) = \sum_{j=1}^{k} \alpha(a_j) + \alpha(b)$, as $n!$ must have as many factors of 2 as $(\prod_{j=1}^{k} a_j!)b!$.

We also have another useful fact following from these three statements. $(n-b)\log n >$

a_j for all $0 \leq j \leq k$. If a_j is small, then the statement follows for large enough n. If $n - b > 1$, this follows as

$$\begin{aligned} n - b &> \alpha(n) - \alpha(b) - \log_2 n \\ &\geq \alpha(a_j) - \log_2 n \\ &> a_j - \log_2 a_j - \log_2 n - 1 \\ &> a_j(1 - \log 2) - \log_2 n \end{aligned}$$

and hence, $a_j < 3(n-b) + \log n < (n-b)\log n$ for large enough n and assuming that the solution is non-trivial. Otherwise, the equation would satisfy the trivial solution, which implies that $n \geq a_j!$ and hence $\log n > a_j$ assuming n is large enough.

We shall also make use of the fact that $\alpha(a_j) > \dfrac{\log a_j!}{\log((n-b)\log n)}$. We will show that this is true:

First, it seen to be true for $1 < a_j < 18$ just by assuming n to be large enough and $n - b \geq 1$, otherwise:

$$\begin{aligned} \alpha(a_j) &\geq a_j - \frac{\log a_j}{\log 2} - 1 \\ &\geq \frac{a_j \log a_j - \frac{(\log a_j)^2}{\log 2} - \log a_j}{\log a_j} \\ &\geq \frac{\log(a_j^{a_j - \frac{\log a_j}{\log 2} - 1})}{\log a_j} \\ &\geq \frac{\log(a_j^{a_j - \frac{\log a_j}{\log 2} - 1})}{\log((n-b)\log n)} \end{aligned}$$

The last step follows as $(n-b)\log n > a_j$. What is left is to show that $a_j^{a_j - \frac{\log a_j}{\log 2} - 1} > a_j!$

$$\begin{aligned} a_j^{a_j - \frac{\log a_j}{\log 2} - 1} &\geq a_j^{a_j - \frac{a_j}{\log(a_j)} + 1} \\ &= a_j^{a_j + 1} e^{-a_j} \end{aligned}$$

$$
\begin{aligned}
&\geq (\frac{a_j}{e})^{a_j}\sqrt{7a_j} \\
&> (\frac{a_j}{e})^{a_j}\sqrt{2\pi a_j e^{\frac{1}{6*2*\pi}}} \\
&> (\frac{a_j}{e})^{a_j}\sqrt{2\pi a_j e^{\frac{1}{6a_j}}} \\
&> a_j!
\end{aligned}
$$

when $a \geq 18 > 2\pi$ by Stirling's approximation.

From this we deduce that

$$
\begin{aligned}
\sum_{j=1}^{k} \alpha(a_j) &> \frac{\sum_{j=1}^{k} \log a_j!}{\log((n-b)\log n)} \\
&= \frac{\log n! - \log b!}{\log((n-b)\log n)} \\
&> \frac{n\log b - b\log b}{\log((n-b)\log n)} \\
&> \frac{(n-b)\log b}{\log((n-b)\log n)}
\end{aligned}
$$

We remark that if $b \geq a_i$ for all $i \leq k$, then the largest prime factor of $n!$ must be the largest prime factor of $b!$. Therefore, there cannot be a prime factor between b and n. A theorem of Baker *et al.*[2] states: If p_n is the nth prime number, then $p_{n+1} - p_n < (p_n)^{0.525+\epsilon}$ for large p_n. It follows that for large enough n, the gap $n - b < b^{0.525+\epsilon}$. However the proof is still valid if we use weaker results such as $p_{n+1} - p_n < (p_n)^{0.9}$.

Since

$$
\begin{aligned}
\sum_{j=1}^{\infty}(\lfloor 2^{-j}n\rfloor - \lfloor 2^{-j}b\rfloor) &= \sum_{j=1}^{\lfloor \log_2 b\rfloor}(\lfloor 2^{-j}n\rfloor - \lfloor 2^{-j}b\rfloor) + \sum_{j=\lfloor \log_2 b\rfloor}^{\lfloor \log_2 n\rfloor}\lfloor 2^{-j}n\rfloor \\
&\leq \sum_{j=1}^{\lfloor \log_2 b\rfloor}(2^{-j}n - (2^{-j}b - 1)) + \frac{n}{b} - 1 \\
&\leq n - b - \sum_{\lfloor \log_2 b\rfloor}^{\infty}(2^{-j}(n-b)) + \log_2 b + \frac{n}{b} - 1
\end{aligned}
$$

$$\begin{aligned} &\leq n - b - \frac{n-b}{b} + \log_2 b + \frac{n}{b} - 1 \\ &< n - b + \log_2 b \end{aligned}$$

we have $\alpha(n) - \alpha(b) < n - b + \log_2 b$

And we get
$n - b + \log_2 b > \sum_{j=1}^{k} \alpha(a_j) > \frac{(n-b)\log b}{\log((n-b)\log n)}$
and thus

$$\begin{aligned} n - b &< \frac{(\log_2 b)\log((n-b)\log n)}{\log b - \log((n-b)\log n)} \\ &< \frac{(\log b)\log(b^{\cdot 9}\log n)}{\log 2(\log b - 2\log\log n)} \\ &< \frac{(\log b)(2\log(b^{\cdot 9}))}{\log 2(\log b - \log\log n)} \\ &< \frac{2(\log b)^2}{\log 2(\log b - \log\log n)} \\ &< \frac{3(\log b)^2}{\log 2(\log b)} \\ &< 5\log b \end{aligned}$$

We again substitute into the equation :

$$\begin{aligned} n - b &< \frac{(\log_2 b)(\log((n-b)\log n)}{\log b - \log((n-b)\log n)} \\ &< \frac{(\log_2 b)\log(15\log b\log n)}{\log b - \log(15\log b\log n)} \\ &< \frac{(\log_2 b)(15\log\log n)}{\log b - 3\log\log b} \\ &< \frac{15(1+\epsilon)\log_2 b(\log\log n)}{(1-\epsilon)\log b} \\ &< 22\log\log n \end{aligned}$$

We repeat this technique :

$$n - b < \frac{(\log_2 b)(\log((n-b)\log n)}{\log b - \log((n-b)\log n)}$$

$$\begin{aligned} &< \frac{(\log_2 b)\log((22\log\log n)\log n)}{\log b - \log(22(\log\log n)\log n)} \\ &< \frac{(\log_2 b)((1+\epsilon)\log\log n)}{\log b - 2\log\log b} \\ &< \frac{(1+\epsilon)\log_2 b(\log\log n)}{(1-\epsilon)\log b} \\ &< \frac{1+\epsilon}{\log 2}\log\log n \end{aligned}$$

So we get our desired result
$n - b < (\frac{1+\epsilon}{\log 2})\log\log n$

4.3 Arithmetic Progressions

In this section, we define $P(n,d) = \prod_{k=1}^{n}(dk)$.

THEOREM: For arbitrary $\epsilon > 0$, there is n_ϵ depending on ϵ such that for all $n > n_\epsilon$, we have $P(m,d) = \prod_{k=0}^{N} P(m_k,d)$ where $dm > dm_N \geq dm_k$, for all k such that $1 \leq k \leq N$. For sake of simplicity, we will define $P = P(m,d)$ and $P_k = P(m_k,d)$

Proof outline: We will define the number c as the greatest prime factor of d. We will first develop a function that counts the number of factors of c in each term. From that we will develop upper and lower bounds for the function and conclude that:

$$d(m - m_N) < \frac{(\log_c(dm_N))\log(d(m-m_N)\log(dm_N))}{\log(dm_N) - \log(d(m-m_N)\log(dm))}$$

From this, we will use the inequalities that we derived to obtain the desired result. When $c > 1$ an integer, we write $\log_c M$ to mean $\frac{\log M}{\log c}$ and use these terms interchangeably.

Proof:

First we define a function $\alpha(n) = \sum_{j=1}^{\infty}\lfloor c^{-j}n\rfloor$, where $\lfloor x\rfloor$ denotes the largest integer less than or equal to x and $\beta(n) = \frac{(c-1)\alpha(n)}{c\log c}$.

A reader should note four things about these functions:

1: When c approaches 1, $\frac{c-1}{\log c}$ approaches 1, so $\beta(n) \to \alpha(n)$ when $c \to 1+$

2: $\beta(n) < n$

3: $\alpha(dm_k)$ counts the number of factors of c in the term P_k.

4: $\alpha(n) = \sum_{j=1}^{\infty} \lfloor c^{-j} n \rfloor = \sum_{j=1}^{\lfloor \log_c n \rfloor} \lfloor c^{-j} n \rfloor > \sum_{j=1}^{\lfloor \log_c n \rfloor} (c^{-j} n - 1) \geq \frac{c}{c-1} n - \frac{\log n}{\log c} - 1$

By Statement 3, $P = (\prod_{k=1}^{N} P_k)$ implies that $\alpha(dm) = \sum_{k=1}^{N} \alpha(dm_k)$, as P must have as many factors of c as $(\prod_{j=1}^{k} P_k)$.

We also have another useful fact following from these three statements: $3d(m - m_N)\log(dm) > dm_k$ for all $0 \leq k \leq N$. If $dm_k < \log(dm)$, then the statement follows trivially. If $dm_k > \log(dm)$, this follows as $d(m - m_N) > \beta(dm) - \beta(dm_N) - \frac{(c-1)}{c} \log_c(dm) \geq \beta(dm_k) - \frac{(c-1)}{c} \log_c(dm) > dm - \frac{(c-1)}{c} \log_c(dm) - \frac{(c-1)}{c} \log_c(dm_k) - 1 > dm - \frac{2(c-1)}{c} \log_c(md)$. Hence, $dm_k < d(m - m_N) + \frac{2(c-1)}{c} \log_c(md) < \frac{3d(c-1)}{c}(m - m_N)\log(dm)$.

We shall also make use of the fact that $\beta(dm_k) > \frac{\log (dm_k)!}{\log((dm - dm_N)\log dm)}$. We will show that this is true:

$$
\begin{aligned}
\beta(dm_j) &\geq dm_j - \frac{(c-1)}{c}\frac{\log dm_j}{\log c} - 1 \\
&\geq \frac{dm_j \log dm_j - \frac{(c-1)}{c \log c}(\log dm_j)^2 - \log dm_j}{\log dm_j} \\
&\geq \frac{\log((dm_j)^{dm_j - \frac{(c-1)}{c \log c}\log dm_j - 1})}{\log dm_j} \\
&\geq \frac{\log((dm_j)^{dm_j - \frac{(c-1)}{c \log c}\frac{\log dm_j}{\log c} - 1})}{\log((m - m_N)\log dm)}
\end{aligned}
$$

The last step follows as $d(m-m_N)\log dm > dm_j$. What is left is to show that $(dm_j)^{dm_j - \frac{\log(dm_j)}{\log c} - 1} > (dm_j)!$

$$\begin{aligned}
(dm_k)^{dm_k - \frac{(c-1)\log_c(dm_k)}{c}} &\geq (\frac{dm_k}{e})^{dm_k}\frac{e^{dm_k}}{(dm_k)^{\log(dm_k)}} \\
&= (\frac{dm_k}{e})^{dm_k}\sqrt{2\pi dm_k}\frac{e^{dm_k}}{\sqrt{2\pi dm_k}(dm_k)^{\log(dm_k)}} \\
&= (\frac{dm_k}{e})^{dm_k}\sqrt{2\pi dm_k}\frac{e^{dm_k-(\log(dm_k))^2-.5}}{\sqrt{2\pi}} \\
&\geq (dm_k)!
\end{aligned}$$

when $dm_k \geq 7$ by Stirling's approximation. For $1 < dm_k < 7$, this can be confirmed by calculation.

From this we deduce that

$$\begin{aligned}
\sum_{j=1}^{N}\beta(dm_j) &> \frac{\sum_{j=1}^{k}\log(dm_j)!}{\log(d(m-m_N)\log(dm))} \\
&= \frac{\log(dm)! - \log(dm)!}{\log(d(m-m_N)\log(dm_N))} \\
&> \frac{dm\log(dm_N) - dm_N\log(dm_N)}{\log(d(m-m_N)\log(dm))} \\
&> \frac{d(m-m_N)\log(dm_N)}{\log(d(m-m_N)\log(dm))}
\end{aligned}$$

We remark that the largest prime factor $\frac{P}{P_N}$ cannot exceed the largest prime factor of $\prod_{k=1}^{N-1} P_k$. This should imply that if $(m-k)$ is prime for any k ranging from 0 to m_N, then $(m-k)|d$. If there is no prime number in this range, we can conclude, as we did in the previous section that $m - m_N < m^{.7}$ or else we can conclude that $m - m_N < d$, so we get the inequality:

$$m - m_N < \frac{(dm_N)^{.8}}{2\log m} < dm_N$$

Since we have $\beta(dm)-\beta(dm_N) < d(m-m_N)+\frac{(c-1)\log_c dm_N}{c} < d(m-m_N)+\log_c(dm_N)$

We get
$d(m-m_N)+\log_c(dm_N) > \beta(dm)+\beta(dm_N) = \sum_{j=1}^{N-1} \beta(dm_j) > \frac{d(m-m_N)\log(dm_N)}{\log(d(m-m_N)\log(dm))}$
and thus
$d(m-m_N) < \frac{(\log_c(dm_N))\log(d(m-m_N)\log(dm))}{\log(dm_N)-\log(d(m-m_N)\log(dm))}$

From the above inequality we have already delivered (namely $m - m_N < \frac{dm_N{}^{.8}}{2\log m} < dm_N$), we have got that

$$\begin{aligned} d(m-m_N) &< \frac{(\log_c(dm_N))\log(d(m-m_N)\log(dm))}{\log(dm_N)-\log(d(m-m_N)\log(dm))} \\ &\leq \frac{2(\log(dm_N))^2}{\log c(\log(dm_N)-\log(d(m_N)^{.8}))} \\ &\leq \frac{2(\log(dm_N))^2}{\log c(\log{(m_N)}^{0.2})} \\ &\leq \frac{2(\log(dm_N))^2}{\log c} \end{aligned}$$

Note: From the above equation, we can also derive the inequality $dm < dm_N+\frac{2(\log m_N)^2}{\log c} < 3dm_N$

We again substitute into the equation :

$$\begin{aligned} d(m-m_N) &< \frac{(\log_c(dm_N))(\log(d(m-m_N)\log(dm)))}{\log(dm_N)-\log(d(m-m_N)\log(dm))} \\ &< \frac{(\log_c(dm_N))(\log(\frac{2(\log(dm_N))^2}{\log c}\log(dm)))}{\log(dm_N)-\log(\frac{2(\log(dm_N))^2}{\log c}\log(dm))} \\ &< \frac{(\log_c(dm_N))(6\log\log(dm))}{\log(dm_N)-6\log\log(dm)} \\ &< \frac{(\log_c(dm_N))(6\log\log(dm))}{\log(dm_N)-6\log\log(3dm_N)} \\ &< \frac{(\log_c(dm_N))(6\log\log(dm))}{0.1\log(dm_N)} \end{aligned}$$

$$
\begin{aligned}
&< \frac{(\log_c(dm_N))(60\log\log(dm_N))}{\log(dm_N)} \\
&< \frac{60\log\log(dm_N)}{\log c}
\end{aligned}
$$

Note: From the above equation, we can also derive the inequality $dm < dm_N + \frac{60\log\log(dm_N)}{\log c} < 6dm_N$ If we assume that $dm > 600$, we can also derive that $dm_N > 100$, and we can repeat the this technique.

$$
\begin{aligned}
d(m-m_N) &< \frac{(\log_c(dm_N))(\log(d(m-m_N)\log(dm)))}{\log(dm_N) - \log(d(m-m_N)\log(dm))} \\
&< \frac{(\log_c(dm_N))(\log(\frac{60\log\log(dm_N)}{\log c}\log(6dm_N)))}{\log(dm_N) - \log(\frac{60\log\log(dm_N)}{\log c}\log(6dm_N))} \\
&< \frac{(\log_c(dm_N))(\log(\frac{60\log\log(dm_N)}{\log c}\log(6dm_N)))}{\log(dm_N) - 0.4\log(dm_N)} \\
&< \frac{(\log_c(dm_N))(\log(\frac{60\log\log(dm_N)}{\log c}\log(6dm_N)))}{0.6\log(dm_N)} \\
&< \frac{(\log_c(dm_N))(\log(\frac{60\log\log(dm_N)}{\log c}\log(6dm_N)))}{0.6\log(dm_N)} \\
&< 9\log\log(dm_N) \\
&< 9\log\log(dm)
\end{aligned}
$$

Note: From the above equation, we can also derive the inequality $dm < dm_N + \frac{9\log\log dm_N}{\log c} < 2dm_N$. On the assumption that m is "large enough," that is the ϵ's will depend on the size of m, we can repeat the technique, and:

$$
\begin{aligned}
d(m-m_N) &< \frac{(\log_c(dm_N))(\log(d(m-m_N)\log(dm))}{\log(dm_N) - \log(d(m-m_N)\log(dm))} \\
&< \frac{(\log_c(dm_N))\log((9\log\log(dm))\log(dm))}{\log(dm_N) - \log(9(\log\log(dm))\log(dm))} \\
&< \frac{(\log_c(dm_N))((1+\epsilon)\log\log(dm))}{\log(dm_N) - 2\log\log(dm_N)} \\
&< \frac{(1+\epsilon)\log_c(dm_N)(\log\log(dm))}{(1-\epsilon)\log(dm_N)}
\end{aligned}
$$

$$< \quad \frac{1+\epsilon}{\log c} \log\log(dm)$$

So we get our desired result:

$$d(m - m_N) < (\frac{1+\epsilon}{\log c}) \log\log dm$$

for large enough d or large enough m.

Appendix A

Appendix

A.1 A Note on Calculus:

Introduction In this section, we give an introduction to the basic tools of analysis that have been omitted from the main part of the thesis for various reasons. We start by presenting an introduction to calculus. Keeping in line with the first section we present a treatment that is loosely based on Edwards' books on calculus [10] [11]. In his early papers, Ramanujan makes reference to Edwards' *Differential Calculus for Beginners* [20] [3], so we have reason to believe that this treatment would resemble his understanding of those tools.

Integrals and Differentials We define the integral of a function$f(Z)$, as $F(Z) = \int_0^Z f(z)dz = \lim_{h \to 0} \sum_{n=0}^{Z/h} hf(nh)$. This function, bluntly, gives the area under the curve $f(z)$ from 0 to z. We define the derivative of a function $F(z)$ with respect to another function $G(z)$ as $f(z) = \frac{dF(z)}{dG(z)} = \lim_{h \to 0} \frac{F(z+h)-F(z)}{G(z+h)-G(z)}$. This function measures the local rate of change of the function $F(z)$ with respect to $G(z)$ at z.

First we prove that $F(Z) = \int_0^Z f(z)dz + F(0)$ if and only if $f(z) = \frac{dF(z)}{dz}$.

$$\begin{aligned} f(z) &= \frac{dF(z)}{dz} \\ &= \lim_{h\to 0}\frac{F(z+h)-F(z)}{z+h-z} \\ &= \lim_{h\to 0}\frac{F(z+h)-F(z)}{h} \end{aligned}$$

Now we multiply h on both sides

$$\begin{aligned} \lim_{h\to 0} hf(z) &= \lim_{h\to 0}(F(z+h)-F(z)) \\ \lim_{h\to 0}\sum_{0}^{z} hf(nh) &= \lim_{h\to 0}\sum_{0}^{z} F(nh)-F((n-1)h) \\ \lim_{h\to 0}\sum_{0}^{z} hf(nh) &= F(z)-F(0) \\ \int_{0}^{z} f(z)dz &= F(z)-F(0) \\ \int_{0}^{z} f(z)dz + F(0) &= F(z) \end{aligned}$$

Manipulation of equations We will now prove some basic theorems that make the above definitions useful.

Theorem 1: Addition theorem : If $F(s) = f(s) + g(s)$, then $\frac{dF(s)}{ds} = \frac{df(s)}{ds} + \frac{dg(s)}{ds}$.

Proof:

$$\begin{aligned} \frac{dF(s)}{ds} &= \lim_{h\to 0}\frac{F(s+h)-F(s)}{(s+h)-s} \\ \frac{dF(s)}{ds} &= \lim_{h\to 0}\frac{f(s+h)+g(s+h)-f(s)-g(s)}{(s+h)-s} \end{aligned}$$

$$\begin{aligned}
\frac{dF(s)}{ds} &= \lim_{h\to 0}\frac{f(s+h)-f(s)}{(s+h)-s}+\lim_{h\to 0}\frac{g(s+h)-g(s)}{(s+h)-s}\\
\frac{dF(s)}{ds} &= \frac{df(s)}{ds}+\frac{dg(s)}{ds}
\end{aligned}$$

Theorem 2: Scaling theorem : If $F(s) = \alpha f(s)$, then $\frac{dF(s)}{ds} = \alpha\frac{df(s)}{ds}$.

Proof:

$$\begin{aligned}
\frac{dF(s)}{ds} &= \lim_{h\to 0}\frac{F(s+h)-F(s)}{(s+h)-s}\\
\frac{dF(s)}{ds} &= \lim_{h\to 0}\frac{\alpha f(s+h)-\alpha f(s)}{(s+h)-s}\\
\frac{dF(s)}{ds} &= \alpha\lim_{h\to 0}\frac{f(s+h)-f(s)}{(s+h)-s}\\
\frac{dF(s)}{ds} &= \alpha\frac{df(s)}{ds}
\end{aligned}$$

Theorem 3: Chain theorem : If $F(s) = f(g(s))$, then $\frac{dF(s)}{ds} = \frac{df(g(s))}{dg(s)}\frac{dg(s)}{ds}$.

Proof:

$$\begin{aligned}
\frac{dF(s)}{ds} &= \lim_{h\to 0}\frac{F(s+h)-F(s)}{(s+h)-s}\\
\frac{dF(s)}{ds} &= \lim_{h\to 0}\frac{f(g(s+h))-f(g(s))}{(s+h)-s}\\
\frac{dF(s)}{ds} &= \lim_{h\to 0}\frac{(f(g(s+h))-f(g(s)))(g(s+h)-g(s))}{((s+h)-s)((g(s+h)-g(s)))}\\
\frac{dF(s)}{ds} &= \lim_{h\to 0}\frac{f(g(s+h))-f(g(s))}{g(s+h)-g(s)}\lim_{h\to 0}\frac{g(s+h)-g(s)}{(s+h)-s}\\
\frac{dF(s)}{ds} &= \frac{df(g(s))}{dg(s)}\frac{dg(s)}{ds}
\end{aligned}$$

Theorem 4: Product theorem : If $F(s) = f(s)g(s)$, then $\frac{dF(s)}{ds} = g(s)\frac{df(s)}{ds} + f(s)\frac{dg(s)}{ds}$.

Proof:

$$\begin{aligned}
\frac{dF(s)}{ds} &= \lim_{h\to 0}\frac{F(s+h)-F(s)}{(s+h)-s} \\
\frac{dF(s)}{ds} &= \lim_{h\to 0}\frac{f(s+h)g(s+h)-f(s)g(s)}{(s+h)-s} \\
\frac{dF(s)}{ds} &= \lim_{h\to 0}\frac{f(s+h)g(s+h)-f(s+h)g(s)+f(s+h)g(s)-f(s)g(s)}{(s+h)-s} \\
\frac{dF(s)}{ds} &= \lim_{h\to 0}\frac{f(s+h)g(s+h)-f(s+h)g(s)}{(s+h)-s} + \lim_{h\to 0}\frac{f(s+h)g(s)-f(s)g(s)}{(s+h)-s} \\
\frac{dF(s)}{ds} &= \lim_{h\to 0} f(s+h)\frac{g(s+h)-g(s)}{(s+h)-s} + g(s)\lim_{h\to 0}\frac{f(s+h)-f(s)}{(s+h)-s} \\
\frac{dF(s)}{ds} &= f(s)\frac{dg(s)}{ds} + g(s)\frac{df(s)}{ds}
\end{aligned}$$

Theorem 5 : $\frac{dx^n}{dx} = nx^{n-1}$ for all integers $n \neq 0$.

Proof: We prove this by induction:

Base Cases: $n = -1, 1$

$$\begin{aligned}
\frac{dx}{dx} &= \lim_{h\to 0}\frac{x+h-x}{x+h-x} = 1 \\
\frac{d\frac{1}{x}}{dx} &= \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{x+h-x} = \lim_{h\to 0}\frac{x-x-h}{hx(x+h)} = \lim_{h\to 0}\frac{-1}{x(x+h)} = -x^{-2}
\end{aligned}$$

Inductive Hypothesis:

For positive $n > 1$, assume that $\frac{dx^{n-1}}{dx} = (n-1)x^{n-2}$

By Product theorem:

$\frac{dx^n}{dx} = x\frac{dx^{n-1}}{dx} + x^{n-1}\frac{dx}{dx} = nx^{n-1}$

For negative $n < -1$, assume that $\frac{dx^n}{dx} = nx^{n-1}$

By Product theorem:

$$\frac{dx^{n-1}}{dx} = \frac{1}{x}\frac{dx^n}{dx} + x^n\frac{dx^{-1}}{dx} = \frac{1}{x}.x^{n-1} + x^n(-x^{-2}) = (n-1)x^{n-2}$$

Theorem 6: Maclaurin's theorem If there exists a series of polynomials $f_k(s) = \sum_{j=0}^{k} a_j x^j$ that converges to $f(s)$, then

$f(s+h) = \sum_{k=0}^{\infty} \frac{f^{(k)}(s)h^k}{k!}$, where $f^{(k)}(s)$ refers to the $'k'$th derivative (the derivative operator applied 'k' times) of $f(s)$.

(Note: A function of this form shall be referred to as an *analytic* function)

Proof: We can denote $f(s+h)$ as $g_s(h) = \sum_{k=0}^{\infty} A_k h^k$, where $A_k = A_k(s)$. Consider $\frac{d^n f(s+h)}{dh^n}$. For $k < n$, $\frac{d^n A_k h^k}{dh^n} = 0$. For $k \geq n$, $\frac{d^n A_k h^k}{dh^n} = \frac{A_k n! h^{n-k}}{k!}$. So we have

$$\frac{d^n f(s+h)}{dh^n} = \sum_{k \geq n} \frac{A_k n! h^{n-k}}{k!}$$

Making use of this, and by setting $h = 0$, we get $A_k = \frac{f^{(k)}(s)}{k!}$, and thus we have proven the required result.

Theorem 7: Taylor's theorem If there is a series of polynomials converges to $f(s)$, then

$f(s) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)s^k}{k!}$, where $f^{(k)}(s)$ refers to the $'k'$th derivative of $f(s)$.

Proof: This follows from Maclaurin's theorem, by setting $h = 0$.

A.2 Calculus involving Complex numbers

A reader should note that by invoking the Scaling and Addition theorems of calculus, our proof requirements for any theorems involving differentiation and integration on analytic functions reduce to the cases of $f(z) = z^n$ for any $n \geq 0$.

Theorem 1: Path Independence of Complex Integrals :

Given an analytic function $f(z)$, the value of $\int_b^a f(z)dz$ is independent of the path from a to b on which the integration takes place.

We can conclude that for any $\{z_k\}$ such that z_k are points that lie on a path that goes from a to b with the condition that $|z_k - z_{k+1}| = h$ in the following limit:

$$\begin{aligned} \lim_{|z_k - z_{k+1}| \to 0} \frac{z_k^{m+1} - z_{k+1}^{m+1}}{|z_k - z_{k+1}| z_k^m} = m+1 &= \lim_{h \to 0} \frac{(z_{k+1} + he^{arg(z_k - z_{k+1})})^{m+1} - z_{k+1}^{m+1}}{hz_k^m} \\ &= m+1 \end{aligned}$$

By constructing equations of the above form using points along a given path such that the points are arbitrarily close together, and using the fact that the right hand side is constant, we can add the numerators and the denominators and get the equation:

$$\begin{aligned} \lim_{h \to 0} \frac{a^{m+1} - b^{m+1}}{h \sum_b^a z_n^m} &= m+1 \\ &\textit{and} \\ \lim_{h \to 0} h \sum_b^a z_n^m &= \frac{a^{m+1} - b^{m+1}}{m+1} \end{aligned}$$

The left hand side corresponds to our definition of an integral. We therefore get that $\int_b^a z^m dz = \frac{a^{m+1} - b^{m+1}}{m+1}$ irrespective of the path taken.

Theorem 2: Cauchy's Integral Theorem : If a function f(z) is analytic and one-valued (i.e.: the output of the function is a single number) inside and on a simple closed contour C, then $\int_C f(s)ds = 0$

Proof: We reproduce the proofs given by Titchmarsh [48].

We break C into a small parts connected by lines parallel to the real axis and the imaginary axis. We shall divide the shapes constructed into two classes: Rectangular (to be refereed to as R_j) and Irregular (to be referred to as D_j). Then:

$$\int_C f(s)ds = \sum \int_{R_j} f(s)ds + \sum \int_{D_j} f(s)ds$$

where each section is integrated in the counter-clockwise direction.
Consider, for example, two sections $WXYZ$ and $ZYTS$ with a common side YZ. The side YZ is described from Y to Z in the first square, and from Z to Y in the second. Hence the two integrals along YZ cancel. So all the integrals cancel, except those which form part of C itself, since these are described

We now use the fact that $f(s)$ is analytic at every point. By an application of Taylor's theorem, we get, provided that $0 < |z - z_0| < \delta = \delta(z_0)$,:

$$\begin{aligned} \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| &< \epsilon \\ |f(z) - f(z_0) - (z - z_0)f'(z_0)| &\leq \epsilon|z - z_0| \end{aligned}$$

In any particular section, R_k or D_j, we can choose its sides so small that $|f(z) - f(z_0) - (z - z_0)f'(z_0)| \leq \epsilon|z - z_0|$ if $0 < |z - z_0| < \delta = \delta(z_0)$, where the circle of radius δ and center z_0 span that section. We shall show that, from here, we can bound the area of the whole region.

Having given ϵ, we can choose the network in such a way that, in every section, R_k or D_j, there is a point z_0 such that $|f(z) - f(z_0) - (z - z_0)f'(z_0)| \leq \epsilon|z - z_0|$ for all z in this section.
Consider one of the rectangles R_k, of perimeter p_k. We have that:

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \phi(z)$$

where $|\phi(z)| \leq \epsilon|z - z_0|$

and

$$\int_{R_k} f(z)dz = \int_{R_k} (f(z_0) + (z - z_0)f'(z_0))dz + \int_{R_k} \phi(z)dz$$

The first integral, on the right is zero (by application of the Path Independence of Complex Integrals, Theorem One of this section). Also, $|\int_{R_k} \phi(z)dz| < \epsilon p_k^2$ since $|z - z_0| < p_k$, and the perimeter of R_k is p_k.

In the case of one of the irregular regions D_j the length is not greater than $s_j + d_j$, where s_j is the length of the straight part of the perimeter and d_j is rest of the perimeter. Hence

$$|\int_{D_j} \phi(z)dz| < \epsilon s_j.(s_j + d_j)$$

Adding all the parts, we obtain

$$|\int_C f(z)dz| < \epsilon \sum (p_k^2 + s_j^2) + \epsilon \sum s_j d_j$$

Now $(\sum(\frac{p_k}{4})^2 + \sum s_j^2)$ is the area of a region which just includes C, and is therefore if $(a, b; \alpha, \beta)$ is a rectangle including C, then $(\sum(\frac{p_k}{4})^2 + \sum s_j^2) < (b - a)(\beta - \alpha)$. $\sum s_n$ is the length of the contour C. Therefore:

$$|\int_C f(z)dz| < \epsilon K(C)$$

where $K(C)$ = the perimeter of C + the area of a rectangle enclosing C is a constant, and $\epsilon > 0$ is arbitrary. Therefore $|\int_C f(z)dz|$ must be zero.

Theorem 4: Cauchy's Residue Theorem : Given a function $F(z)$, that can be approximated by the series $F(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, within a circle, C of radius r, the value of $\int\limits_C F(z)dz = 0$ if the function is bounded. If the function goes to infinity at a single point, 0, the value of $\int\limits_C F(z)dz = a_{-1}$. (Note: A function of this form shall be referred to as a *regular* function)

Proof: The following proof is a standard proof of this theorem, and we do not attribute it to any particular source.

$$\begin{aligned}
\int\limits_C F(z) &= \int\limits_C \sum_{n=-\infty}^{\infty} a_n z^n dz \\
&= \sum_{n=-\infty}^{\infty} a_n \int\limits_C z^n dz \quad \text{(scaling and addition theorems)} \\
&= \sum_{n=-\infty}^{\infty} a_n \int\limits_0^{2\pi} \frac{dz}{d\theta} r^n e^{in\theta} d\theta \quad \text{(set z = e}^{i\theta}\text{)} \\
&= \sum_{n=-\infty}^{\infty} a_n \int\limits_0^{2\pi} \frac{dre^{in\theta}}{d\theta} r^n e^{in\theta} d\theta \quad \text{(Chain theorem)} \\
&= \sum_{n=-\infty}^{\infty} a_n \int\limits_0^{2\pi} (r\frac{de^{i\theta}}{d\theta} + e^{i\theta}\frac{dr}{d\theta}) r^n e^{in\theta} d\theta \quad \text{(Product theorem)} \\
&= \sum_{n=-\infty}^{\infty} a_n \int\limits_0^{2\pi} \frac{de^{i\theta}}{d\theta} r^{n+1} e^{in\theta} d\theta \quad \text{(since r is constant)} \\
&= \sum_{n=-\infty}^{\infty} a_n \int\limits_0^{2\pi} ir^{n+1} e^{i(n+1)\theta} d\theta \\
&= \sum_{n=-\infty, n\neq -1}^{\infty} a_n ir^{n+1}(e^{2i\pi(n+1)} - e^0) + a_{-1} \int\limits_0^{2\pi} ie^0 d\theta \\
&= \sum_{n=-\infty, n\neq -1}^{\infty} a_n ir^{n+1}(e^0 - e^0) + a_{-1} \int\limits_0^{2\pi} ie^0 d\theta \quad \text{(by Euler's Formula)} \\
&= \sum_{n=-\infty, n\neq -1}^{\infty} a_n ir^{n+1}(0) + 2i\pi a_{-1} \\
&= 2i\pi a_{-1}
\end{aligned}$$

Note if there is a *pole* (a single isolated point where $|F(z)|$ goes to ∞) at $z = z_0$ then one can replace z with $|z - z_0|$. The value for a_{-1} is called the *residue*.

Theorem 5: Perron's formula : *Let* $y > 0$. *Then* $\frac{1}{2i\pi}\int\limits_{2-i\infty}^{2+i\infty} y^s \frac{ds}{s} = \begin{cases} 0 & if \quad y < 1 \\ \frac{1}{2} & if \quad y = 1 \\ 1 & if \quad y > 1 \end{cases}$

Proof: This proof is the one presented in [39]. For an interesting discussion please see [24].

First we will consider the case of $y > 1$. Consider a rectangle with corners at $c \pm iR$ and $d \pm iR$ where d is negative. Cauchy's Integral theorem tells us that

$$\begin{aligned} \frac{1}{2\pi i}\int\limits_{c-iR}^{c+iR} \frac{y^s}{s} ds - 1 &= \frac{1}{2\pi i}\Big(\int\limits_{c-iR}^{d-iR} + \int\limits_{d-iR}^{d+iR} + \int\limits_{d+iR}^{c+iR}\Big)\frac{y^s}{s} ds \\ &= I_1 + I_2 + I_3 \end{aligned}$$

The -1 term comes from the fact that $\frac{y^s}{s}$ has a pole at 0 with residue 1. We also get a bound on $|I_2| \le \frac{2Ry^d}{2\pi d}$. As d becomes a large negative number, this tends to 0.

$$\begin{aligned} \frac{1}{2\pi i}\int\limits_{c-iR}^{c+iR} \frac{y^s}{s} ds - 1 &= I_1 + I_3 + O(\frac{Ry^d}{d}) \\ &= \frac{2}{\pi} Im(\int\limits_{\infty}^{c} \frac{y^{iR+\sigma}}{iR+\sigma} d\sigma) + O(\frac{Ry^d}{d}) \\ &= \frac{2}{\pi} Im(\int\limits_{\infty}^{c} \frac{y^\sigma e^{iR\log y}}{iR+\sigma} d\sigma) + O(\frac{Ry^d}{d}) \\ &= \frac{2}{\pi} Im(\int\limits_{\infty}^{c} \frac{y^\sigma(\cos(R\log y) + i\sin(R\log y))}{iR+\sigma} d\sigma) + O(\frac{Ry^d}{d}) \\ &= \frac{2}{\pi} Im(\int\limits_{\infty}^{c} \frac{y^\sigma(\cos(R\log y) + i\sin(R\log y))(\sigma - iR)}{R^2+\sigma^2} d\sigma) + O(\frac{Ry^d}{d}) \\ &= \frac{2}{\pi}(\int\limits_{-\infty}^{c} \frac{y^\sigma}{R^2+\sigma^2}(\sigma\sin(R\log y) - R\cos(R\log y)) d\sigma) + O(\frac{Ry^d}{d}) \\ &\le \int\limits_{-\infty}^{\sigma} \frac{R\log y}{2\pi R\sigma} y^\sigma + \frac{y^c R}{\pi(\sigma^2+R^2)} d\sigma + O(\frac{Ry^d}{d}) \\ &\le \frac{y^c}{\pi}(\frac{\pi+1}{2} + \tan^{-1}(\frac{c}{R})) + O(\frac{Ry^d}{d}) \end{aligned}$$

From the first expression for $I_1 + I_3$, we get the upper bound

$$\frac{1}{\pi}\int_{-\infty}^{c}\frac{y^\sigma}{R}d\sigma = \frac{y^c}{\pi R\log y}$$

and hence the case of $y > 1$ is proven.

In the case of $y < 1$, we use a similar argument.

Again, we get a bound on $|I_2| \leq \frac{2Ry^d}{2\pi d}$. As d becomes a large positive number, this tends to 0.

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{y^s}{s}ds &= \frac{1}{2\pi i}\left(\int_{c-iR}^{d-iR} + \int_{d-iR}^{d+iR} + \int_{d+iR}^{c+iR}\right)\frac{y^s}{s}ds \\
&= I_1 + I_2 + I_3 \\
&= I_1 + I_3 + O\left(\frac{Ry^d}{d}\right) \\
&= \frac{1}{\pi}2Im\left(\int_{\infty}^{c}\frac{y^{\sigma-iR}}{\sigma - iR}d\sigma\right) + O\left(\frac{Ry^d}{d}\right) \\
&= \frac{1}{\pi}2Im\left(\int_{-\infty}^{c}\frac{y^\sigma}{\sigma - iR}(\sigma\sin(-R\log y) - R\cos(-R\log y))d\sigma\right) + O\left(\frac{Ry^d}{d}\right) \\
&\leq \int_{-\infty}^{c}\frac{1}{\pi R}y^\sigma + \frac{y^c R}{\pi(\sigma^2 + R^2)}d\sigma + O\left(\frac{Ry^d}{d}\right) \\
&\leq \frac{y^c}{\pi}\left(\frac{\pi + 1}{2} + \tan^{-1}\left(\frac{c}{R}\right)\right) + O\left(\frac{Ry^d}{d}\right)
\end{aligned}$$

From the first expression for $I_1 + I_3$, we get the upper bound

$$\frac{1}{\pi}\int_{-\infty}^{c}\frac{y^\sigma}{R}d\sigma = \frac{y^c}{\pi R\log y}$$

and hence the case of $y > 1$ is proven.

In the case of $y = 1$:

$$\begin{aligned}
\frac{1}{2\pi i}\int\limits_{c-iR}^{c+iR}\frac{y^s}{s}ds &= \frac{1}{2\pi i}\int\limits_{c-iR}^{c+iR}\frac{1}{s}ds \\
&= \frac{1}{2\pi i}(\ln(c+iR)-\ln(c-iR)) \\
&= \frac{1}{2\pi i}(\ln(\frac{c+iR}{c-iR})) \\
&= \frac{1}{2\pi i}(\ln(\frac{|c+iR|e^{iarg(c+iR)}}{|c-iR|e^{iarg(c-iR)}})) \\
&= \frac{1}{2\pi i}(\ln(\frac{|c+iR|e^{iarg(c+iR)}}{|c+iR|e^{-iarg(c+iR)}})) \\
&= \frac{1}{2\pi i}(\ln(e^{2iarg(c+iR)})) \\
&= \frac{1}{2\pi i}(2iarg(c+iR)) \\
&= \frac{\frac{\pi}{2}-tan^{-1}(\frac{c}{R})}{\pi} \\
&= \frac{1}{2}+O(tan^{-1}(\frac{c}{R}))
\end{aligned}$$

and hence the case of $y = 1$ is proven, and the proof is completed.

A.3 A remark on the Möbius function

Claim: If $\mu(n)$ is as defined in Chapter 2 as the Möbius function and $\lambda(n)$ is the function described as the Liouville function, then any bound of $\sum\limits_{0<n<x}\mu(n)$ of the form $O(x^{\alpha})$ holds for $\sum\limits_{0<n<x}\lambda(n)$ and vice versa.

Proof:

$$\sum_{n<x}\mu(n)=O(x^{\alpha}) \text{ implies } \sum_{n<x}\lambda(n)=O(\sum_{k=1}^{\infty}\sum_{n<\sqrt[k]{x}}\mu(n))=O(\sum_{k=1}^{\infty}\sqrt[k]{x^{\alpha}})=O(x^{\alpha})$$

and

$$\sum_{n<x}\lambda(n)=O(x^{\alpha}) \text{ implies } \sum_{k=1}^{\infty}\mu(n)=O(\sum_{k=1}^{\infty}\sum_{n<\sqrt[k]{x}}\mu(n))=O(\sum_{k=1}^{\infty}\sqrt[k]{x^{\alpha}})=O(x^{\alpha}).$$

This concludes the proof. Below is a table with values computed for both functions.

Table A.1

N	$\omega(N)$	$\Omega(N)$	$\mu(N)$	$\sum\mu(N)$	$\lambda(N)$	$\sum\lambda(N)$
0	0	0	1	1	1	1
1	0	0	1	2	1	2
2	1	1	-1	1	-1	1
3	1	1	-1	0	-1	0
4	1	0	-1	-1	0	0
5	1	1	-1	-2	-1	-1
6	2	2	1	-1	1	0
7	1	1	-1	-2	-1	-1
8	1	0	-1	-3	0	-1
9	1	0	-1	-4	0	-1
10	2	2	1	-3	1	0
11	1	1	-1	-4	-1	-1
12	2	1	1	-3	-1	-2
13	1	1	-1	-4	-1	-3
14	2	2	1	-3	1	-2
15	2	2	1	-2	1	-1
16	1	0	-1	-3	0	-1
17	1	1	-1	-4	-1	-2
18	2	1	1	-3	-1	-3
19	1	1	-1	-4	-1	-4
20	2	1	1	-3	-1	-5
21	2	2	1	-2	1	-4
22	2	2	1	-1	1	-3
23	1	1	-1	-2	-1	-4
24	2	1	1	-1	-1	-5
25	1	0	-1	-2	0	-5
26	2	2	1	-1	1	-4
27	1	0	-1	-2	0	-4
28	2	1	1	-1	-1	-5
29	1	1	-1	-2	-1	-6
30	3	3	-1	-3	-1	-7
31	1	1	-1	-4	-1	-8
32	1	0	-1	-5	0	-8
33	2	2	1	-4	1	-7
34	2	2	1	-3	1	-6
35	2	2	1	-2	1	-5
36	2	0	1	-1	0	-5
37	1	1	-1	-2	-1	-6
38	2	2	1	-1	1	-5
39	2	2	1	0	1	-4
40	2	1	1	1	-1	-5
41	1	1	-1	0	-1	-6
42	3	3	-1	-1	-1	-7
43	1	1	-1	-2	-1	-8
44	2	1	1	-1	-1	-9
45	2	1	1	0	-1	-10

Continued on Next Page...

Table A.1

N	$\omega(N)$	$\Omega(N)$	$\mu(N)$	$\sum\mu(N)$	$\lambda(N)$	$\sum\lambda(N)$
46	2	2	1	1	1	-9
47	1	1	-1	0	-1	-10
48	2	1	1	1	-1	-11
49	1	0	-1	0	0	-11
50	2	1	1	1	-1	-12
51	2	2	1	2	1	-11
52	2	1	1	3	-1	-12
53	1	1	-1	2	-1	-13
54	2	1	1	3	-1	-14
55	2	2	1	4	1	-13
56	2	1	1	5	-1	-14
57	2	2	1	6	1	-13
58	2	2	1	7	1	-12
59	1	1	-1	6	-1	-13
60	3	2	-1	5	1	-12
61	1	1	-1	4	-1	-13
62	2	2	1	5	1	-12
63	2	1	1	6	-1	-13
64	1	0	-1	5	0	-13
65	2	2	1	6	1	-12
66	3	3	-1	5	-1	-13
67	1	1	-1	4	-1	-14
68	2	1	1	5	-1	-15
69	2	2	1	6	1	-14
70	3	3	-1	5	-1	-15
71	1	1	-1	4	-1	-16
72	2	0	1	5	0	-16
73	1	1	-1	4	-1	-17
74	2	2	1	5	1	-16
75	2	1	1	6	-1	-17
76	2	1	1	7	-1	-18
77	2	2	1	8	1	-17
78	3	3	-1	7	-1	-18
79	1	1	-1	6	-1	-19
80	2	1	1	7	-1	-20
81	1	0	-1	6	0	-20
82	2	2	1	7	1	-19
83	1	1	-1	6	-1	-20
84	3	2	-1	5	1	-19
85	2	2	1	6	1	-18
86	2	2	1	7	1	-17
87	2	2	1	8	1	-16
88	2	1	1	9	-1	-17
89	1	1	-1	8	-1	-18
90	3	2	-1	7	1	-17
91	2	2	1	8	1	-16
92	2	1	1	9	-1	-17
93	2	2	1	10	1	-16
94	2	2	1	11	1	-15
95	2	2	1	12	1	-14
96	2	1	1	13	-1	-15
97	1	1	-1	12	-1	-16
98	2	1	1	13	-1	-17
99	2	1	1	14	-1	-18
100	2	0	1	15	0	-18

A.4 Three Theorems of Analytic number theory and an application

The following three theorems will be used to prove that Lindelöf hypothesis is a consequence of the Riemann hypothesis. We reproduce the proofs given by Titchmarsh [49].

Theorem 1: Maximum Modulus Principle : Given that $f(z)$ is an analytic complex function, if $|f(z)| \leq M$ on the boundary of a region C then $|f(z)| < M$ at all interior points of C, unless $f(z)$ is a constant function.

Proof: We present the proof given in [48]:
Assuming that $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{-\infty}^{\infty} |a_n|^2 r^{2n} \leq |f(0)|^2 \leq |a_0|^2$. This implies that $f(z) = a_0$ is a constant function.

Theorem 2: Hadamard's three circle theorem : Let $f(s)$ be an one-valued complex analytic function (i.e.: an analytic function that takes complex numbers as an input and gives a complex number as an output) in a region C. Let $0 < r_1 < r_2 < r_3$, and choose z_0 such that the circle with center z_0 and radius r_3 is completely contained in C. Let M_1, M_2, and M_3 be the maximum of value of $f(s)$ on the circles with center z_0 and radii r_1, r_2, and r_3. Then:

$$M_2^{\log r_3 - \log r_1} \leq M_1^{\log r_3 - \log r_2} M_3^{\log r_2 - \log r_1}$$

Proof: Consider the function $F(s) = s^c f(s)$, where c is a constant, not yet defined. We can say that $F(s)$ is analytic in the circle with center z_0 and radius r_3. In the ring that is bounded by the radii r_1 and r_3, by the Maximum Modulus Principle, the maximum of $F(s) \leq \max(r_1^c M_1, r_3^c M_3)$. Therefore, the maximum on the circle with radius r_2 is $\max((\frac{r_1}{r_2})^c M_1, (\frac{r_3}{r_2})^c M_3)$. c is unspecified, so by choosing $c = -\dfrac{\log(M_3) - \log(M_1)}{\log(r_3) - \log(r_1)}$, we get $M_2 \leq (\frac{r_1}{r_2}) M_1$ and therefore

$$M_2^{\log(r_3) - \log(r_1)} \quad \leq \quad (\frac{r_2}{r_1})^{\log(M_3) - \log(M_1)} M_1^{\log(r_3) - \log(r_1)}$$

$$\leq \quad M_1^{\log r_3 - \log r_2} M_3^{\log r_2 - \log r_1}$$

Theorem 3: Borel-Carathéodory Theorem : Let $f(z)$ be an analytic function regular for $|z| \leq R$, and let $M(r)$ and $A(r)$ denote the maxima of $|f(z)|$ and $Re\{f(z)\}$ respectively on $|z| = r$. Then for $0 < r < R$

$$M(r) \leq \frac{2r}{R-r} A(R) + \frac{R+r}{R-r} |f(0)|$$

Proof: The result is obvious if $f(z)$ is a constant. If $f(z)$ is not constant, suppose first that f(0) = 0. Then $A(R) > A(0) = 0$.

We define $\phi(z) = \frac{f(z)}{2A(R)-f(z)}$. $\phi(z)$ is regular for $|z| \leq R$, since the real part of the denominator does not vanish; $\phi(0) = 0$; and, if $f(z) = u + iv$,

$$|\phi(z)|^2 = \frac{u^2 + v^2}{(2A(R) - u)^2 + v^2} \leq 1$$

since $-2A(R) + u \leq u \leq A(R) \leq 2A(R) - u$.

By Maximum Modulus Principle, for all z such that $|z - z_0| < R$, $|\frac{\phi(z)}{z}| < \frac{M}{R}$, where M is the maximum of $\phi(z)$ on the circle with radius R since $|z| = R$ on that circle. Since $|\phi(z)| \leq 1$, we know that $M \leq 1$ so $|\frac{\phi(z)}{z}| < \frac{1}{R}$, and we get that $|\phi(z)| < \frac{|z|}{R} = \frac{r}{R}$. Hence $|f(z)| = |\frac{2A(R)\phi(z)}{1+\phi(z)}| \leq \frac{2A(R)r}{R-r}$. We apply the result already obtained to $f(z) - f(0)$:
$|f(z) - f(0)| \leq \frac{2r}{R-r} \max Re(f(z) - f(0)) \leq \frac{2r}{R-r}(A(R) + |f(0)|)$

and the result again follows. If $A(R) \geq 0$, we deduce
$M(r) \leq \frac{R+r}{R-r}(A(R) + |f(0)|)$.
We can obtain similar results with $-f(z)$, or with $\pm i f(z)$, by replacing $A(r)$ with $\min Re(f(z))$, $\max Im(f(z))$, or $\min Imf(z)$ respectively.

A.5 Lindelöf hypothesis: A consequence of the Riemann hypothesis

Theorem : The condition $\zeta(s) = \zeta(\sigma+it) = 0$ only if $\sigma \le \frac{1}{2}$, implies $\zeta(\sigma+it) = O_\epsilon(t^\epsilon)$ for $\frac{1}{2} + \epsilon < \sigma \le 1$ and $t > 10$.

Proof: We reproduce the proof given by Titchmarsh [49]. We shall consider the theorem for any complex number $s = \sigma + it$. Apply the Borel-Carathéodory theorem to $\log\zeta(s)$ and the circles with center $2+it$ and radii $\frac{3}{2} - \frac{1}{2}\delta$ and $\frac{3}{2} - \delta$, where $\delta = \delta(\sigma) = 1 - \sigma$. The circles will be denoted C_1 and C_2 respectively. On C_2, $Re(\log(\zeta(s))) = \log|\zeta(s)| < k\log t$, where k is a constant, and $t = Im(s)$.

Hence, on C_1,

$$\begin{aligned} |\log\zeta(s)| &\le \frac{3-2\delta}{\frac{1}{2}\delta}k\log t + \frac{3-\frac{3}{2}\delta}{\frac{1}{2}\delta}|\log|\zeta(2+it)|| \\ &< \frac{A}{\delta}\log t \end{aligned}$$

where A is a constant.

Consider the circles of common center $\log\log(t+it)$, and radii $\log\log(t-1-\nu)$, $\log\log(t-\sigma)$ and $\log\log t - \frac{1}{2} - \delta$. The radii will be denoted r_1, r_2, and r_3 respectively. The circles will be called C_1, C_2 and C_3 and their maximum value will be referred to as M_1, M_2, and M_3. By Hadamard's three circle theorem, we have $M_2 \le M_1^{1-a}M_3^a$, where

$$\begin{aligned} a &= \frac{\log r_2 - \log r_1}{\log r_3 - \log r_1} \\ &= \log(1 + \frac{1+\nu-\sigma}{\log\log t - 1 - \nu})/\log(1 + \frac{\frac{1}{2}+\nu-\delta}{\log\log t - 1 - \nu}) \\ &= \frac{1+\nu-\sigma}{\frac{1}{2}+\nu-\delta} + O(\frac{1}{\log\log t}) \\ &= 2 - 2\sigma + O(\delta) + O(\nu) + O(\frac{1}{\log\log t}) \end{aligned}$$

We can use the result of our application of the Borel-Carathéodory theorem to justify $M_3 < \frac{A}{\delta}$, and since $\log\zeta(s) < \zeta(s)$, we have that $M_1 < \max_{s\ge 1+\nu}|\zeta(1+\nu)| < \frac{A}{\nu}$.

and we conclude that

$$|\log\zeta(\sigma+it)| < (\frac{A}{\nu})^{1-a}(\frac{A\log t}{\delta})^a < \frac{A(\log t)^{2-2\sigma+O(\delta)+O(nu)+O(1/\log\log t)}}{\nu^{1-a}\delta^a}$$

We can take $\frac{1}{\delta} = \frac{1}{\nu} = \log\log t$. This would make the right hand side $\frac{k_0}{A}(\log t)^{2-2\sigma}$, where $k_0 = (\log t)^{O(1/\log\log t)} =$ a constant.

We now have that $|\log\zeta(\sigma+it)| < O(\log\log t(\log t)^{2-2\sigma})$ for $\frac{1}{2} + \frac{1}{\log\log t} \leq \sigma \leq 1$, and we have our desired result, namely, for all $\sigma > \frac{1}{2}$,

$$\begin{aligned} -\epsilon\log t &< \log|\zeta(\sigma+it)| < \epsilon\log t \\ &\mathit{or} \\ \zeta(s) &= O(t^\epsilon) \\ &\mathit{and} \\ \frac{1}{\zeta(s)} &= O(t^\epsilon) \end{aligned}$$

Publications based on this Thesis

1. K. G. Bhat and K. Ramachandra, *A remark on factorials that are products of factorials*, Matematicheskie Zametki, Karatsuba memorial edition, **88** Numbers 3-4, (2010), p. 317-320, DOI: 10.1134/S0001434610090038.

2. K. G. Bhat, K. Ramachandra, and P.G. Vaidya, *An Introduction to Trigonometry (A new outlook)*, The Mathematics Student, **78** Numbers -14, (2009), p. 187-207.

3. K. G. Bhat and K. Ramachandra, *A remark on a theorem of A.E.Ingham*, Hardy-Ramanujan Journal. **29**, (2006) p.37-43.

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