Multiplexing of discrete chaotic signals in presence of noise

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Multiplexing of discrete chaotic signals in presence of noise is investigated. The existing methods are based on chaotic synchronization, which is susceptible to noise, precision limitations, and requires more iterates. Furthermore, most of these methods fail for multiplexing more than two discrete chaotic signals. We propose novel methods to multiplex multiple discrete chaotic signals based on the principle of symbolic sequence invariance in presence of noise and finite precision implementation of finding the initial condition of an arbitrarily long symbolic sequence of a chaotic map. Our methods work for single precision and as less as 35 iterates. For two signals, our method is robust up to 50% noise level. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3157183]

Multiplexing of signals is a very important requirement in multiuser communication systems. Consider the scenario where there are multiple signals from multiple senders to be transmitted to multiple receivers, but there exists only one communication channel that can transmit only one scalar signal at a time. In such a scenario, it would be beneficial if all the signals are "added" in a special way to create a single composite scalar signal for transmission across the communication channel. This single composite scalar signal is "separated" to the respective signals in a lossless or near lossless fashion at the other end of the channel. However, there is noise which is invariably added at the channel. This scenario can also occur in transmission of neuronal signals from different parts of the brain to various parts of the body through a single pathway. In this work, we investigate multiplexing of discrete chaotic signals in presence of noise.

I. INTRODUCTION

For linear communication systems, standard ways such as frequency division multiplexing (different signals are allocated to different parts of the frequency spectrum) and time division multiplexing (different signals are allocated to different time slots for transmission) are used to increase the information capacity of the channel.[1] Nonlinear chaotic oscillators are increasingly being used in communications since they offer a potential advantage over conventional classical methods in terms of noise performance.[2] Multiuser chaotic communication has become a hot topic of research in recent times.[3] It is also potentially useful in spectrum-spreading communication systems. Hence there is a need for multiplexing chaotic signals.

The existing methods of multiplexing discrete chaotic signals are based on chaotic synchronization, which is susceptible to noise and precision limitations. Furthermore, most of these methods fail for multiplexing more than two discrete chaotic signals. We propose novel methods to multiplex multiple discrete chaotic signals in presence of noise.

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II. EXISTING WORKS AND THEIR LIMITATIONS

There has already been some work in multiplexing chaotic signals. For the first time in 1996, multiplexing of chaos using chaotic synchronization was investigated in a simple map and an electronic circuit model by Tsimring and Sushchik.[4] Liu and Davis[5] used a scalar signal to simultaneously synchronize two different pairs of chaotic oscillators. They called this method as dual synchronization. However, there are several limitations of this method. They derive a condition for dual synchronization which holds only for certain discrete chaotic signals (maps) and for certain values of the coupling coefficients. The notable omission is the binary map (Bernoulli shift). They show that the binary map does not satisfy the condition for dual synchronization for any value of the coupling coefficients. Thus, chaotic signals from the binary map cannot be multiplexed by their method. Another limitation is that their method can only work with two chaotic signals. It is not known whether their method can be extended to multiple signals (more than two) from different maps.

Dual synchronization has been implemented in electronic circuits.[6–9] Blakely and Corron[9] use a symbolic dynamics-based approach to multiplex two pairs of low frequency chaotic electronic circuits that produce Rössler-like oscillations by synchronization. They note that high quality multiplexed synchronization is needed for successful recovery of transmitted information and even a small synchronization error can lead to bit errors.

Most of the early developments were only for multiplexing two chaotic signals. Although Tsimring and Sushchik[4] propose a way to synchronize multiple (more than two) pairs of discrete chaotic signals, it has some serious limitations as follows.

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(1) The response variables at the receiver need to be globally coupled and hence are not independent. This may be a limitation for practical communication applications.
(2) The method heavily depends on optimal choice of global coupling at the receiver and also on the type of nonlinearity (only works for certain kinds of maps).
(3) For multiplexing more than two discrete chaotic signals, two scalar variables need to be transmitted—the real and imaginary parts of the sum of dynamical variables in order to synchronize more than two pairs of maps. This means that the channel needs to carry two scalar signals whereas traditional multiplexing schemes allow only one scalar signal on the channel.
(4) As the authors point out, even a small amount of noise is able to destroy synchronization in spite of superstable synchronization. They demonstrate that white noise of magnitude 10^{-4} is sufficient to break down multiplexing for two pairs of coupled tent maps. With increasing number of maps, the sensitivity to noise only worsens.
(5) The method is highly sensitive to round off errors. With single precision for the signal values, only 13 maps could be synchronized.
(6) The empirical results show that it requires over 10 000 iterations to synchronize.
(7) The method is plagued with intermittency behavior, i.e., bursts of complete desynchronization which can be quite long before synchronization can be restored.
(8) The method as proposed cannot be utilized to transmit a nonchaotic information carrying signal along with chaotic signals. However, the authors do claim that it is possible with certain modifications.

There has been more work in multiplexing chaotic signals from continuous chaotic systems (flows). Liu and Davis[5] extend their work to multiplex signals from delay-differential equations. Further progress has been made by Ning *et al.*[10] who extend Liu's method for three-dimensional continuous chaotic systems (Lorenz and Rössler systems). This has been further improved by Salarieh and Shahrokhi[11] who make use of a time-varying output feedback strategy to achieve dual synchronization without the need for all the master states (only a linear combination of the master states are enough).

The next important development happened very recently with Salarieh and Shahrokhi[12] who succeeded in multiplexing more than two *continuous* chaotic signals using chaotic synchronization via output feedback strategy. They derive a necessary condition for multisynchronization and demonstrate the algorithm for the Chen–Lorenz–Rössler and the Duffing–Van der Pol continuous time chaotic dynamical systems. However, these methods are not yet known to work for multiple *discrete* chaotic signals. As already noted, a serious limitation is that chaotic synchronization is susceptible to noise. Even 1% of noise results in a synchronization error of 4%, as reported by Liu and Davis.[5]

Vaidya's method[13] which multiplexes more than two discrete chaotic signals remains as the latest development on multiplexing chaotic signals from discrete chaotic systems [one-dimensional (1D) maps]. Vaidya's method does not use chaotic synchronization and is fundamentally different from the previous approaches. We will describe the method and its drawbacks in Sec. III C since we are going to use some of the ideas from this method to improve on it. In principle, it is possible to extend the methods that are proposed in this paper to flows and to higher dimensional chaotic dynamical systems, but these will not be pursued here. Since Liu and Davis' method does not work for the standard binary map, we shall consider chaotic signals from the standard binary map with randomly chosen initial conditions.

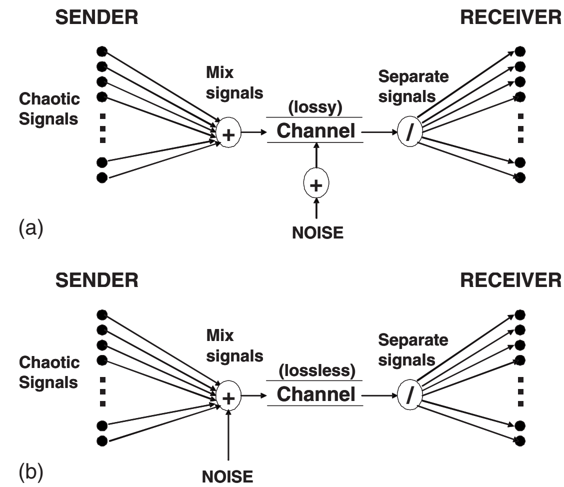

FIG. 1. Multiplexing of chaotic signals in presence of noise. (a) Scenario 1: Channel is lossy, noise is additive and limited in magnitude (Methods 1 and 2). Applications: multi-user communications. (b) Scenario 2: Channel is lossless, but noise which has the same magnitude as the signal is added in a special way at the sender (Method 3). Applications: steganography (data hiding).

A. New approach

Our approach considers two different scenarios, as shown in Fig. 1. In scenario 1, the communication channel is noisy and there is no control on noise that is added during transmission of the signal [Fig. 1(a)]. However, it shall be assumed that the magnitude of noise is limited (we shall give conditions on the magnitude of noise that is allowed by our methods). Noise is uniformly distributed (white noise) and the signals are chaotic. This corresponds to multiuser communication systems.

In scenario 2 [Fig. 1(b)], the communication channel is lossless, but noise is added to the sender. Noise is assumed to be of the same magnitude as the chaotic signals and uniformly distributed. But the way in which noise is added is under control. This scenario corresponds to steganography (data hiding)[18] or cryptographic applications where noise could be the "payload" (secret information) to be secretly transmitted.

Recently, Vaidya[13] suggested a novel multiplexing algorithm for 1D discrete chaotic signals in presence of noise. We shall call this as method 1 and review it briefly and list some of its limitations. A new method (method 2) will be proposed which overcomes some of the limitations of method 1. Both these methods are solutions to scenario 1. A novel method (method 3) is proposed for scenario 2.

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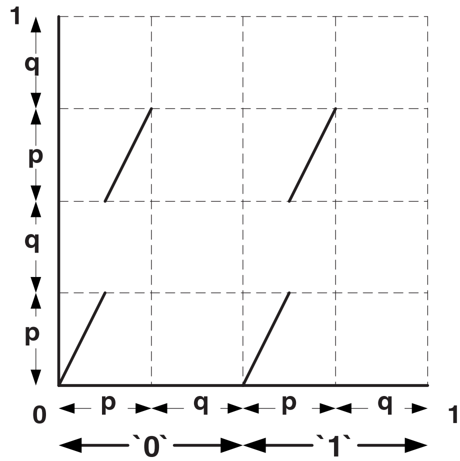

FIG. 2. T_{noiseres}: noise-resistant binary map.

III. METHOD 1: VAIDYA'S NOISE-RESISTANT MAP

Vaidya's method[13] is a solution for multiplexing chaotic signals in presence of noise (scenario 1), which does not make use of chaotic synchronization like previous approaches. Vaidya proposes a noise-resistant version of the tent map. Since we are going to deal mainly with the standard binary map, a minor modification leads to a noise-resistant version of the binary map. It is given by the following set of equations:

$$y = 2x, \quad \text{if} \quad 0 \le x < \frac{p}{2},$$

$$= 2x + q, \quad \text{if} \quad \frac{p}{2} \le x < p,$$

$$= 0, \quad \text{if} \quad p \le x < p + q,$$

$$= 2x - 2(p+q), \quad \text{if} \quad p + q \le x < \frac{3p}{2} + q,$$

$$= 2x - (2p+q), \quad \text{if} \quad \frac{3p}{2} + q \le x < 2p + q,$$

$$= 0, \quad \text{if} \quad 2p + q \le x < 1.$$

Figure 2 depicts the noise-resistant binary map (denoted by T_{noiseres}). Vaidya establishes that there exists conjugacy between the ordinary binary map and T_{noiseres}. Given any chaotic signal on the binary map (a chaotic signal is trajectory on the map for a given initial condition), one can find the equivalent signal on T_{noiseres}. $p(0 < p \le 0.5)$ and $q(0 \le q < 0.5)$ can be chosen such that $2p+2q=1$.

Next, we define the symbolic sequence as follows:

$$S(x_i) = 0, \quad \text{if} \quad 0 \le x_i < p + q,$$

$$= 1, \quad \text{if} \quad p + q \le x_i < 1.$$

Here, $X = \{x_i\}_{i=1}^{i=m}$ is the chaotic trajectory (or chaotic signal) starting from an initial condition x_1. $S(X)$ denotes the symbolic sequence for the entire trajectory.

A. Noise resistance

For any given chaotic signal X on T_{noiseres}, if noise $N = \{n_i\}_{i=1}^{i=m}$ is added such that each n_i satisfies $0 \le n_i < q$, then it can be seen that the symbolic sequence remains unchanged,

$$S(X) = S(X + N). \tag{1}$$

The signal X is transmitted to the sender and the resulting signal $Z = X + N$ is received at the receiver. However, because of the above property of *symbolic* sequence invariance, we can compute $S(X)(=S(Z))$ and iterating backward on the map, we can find the initial condition x_1. Knowing x_1, we can easily compute $x_2, \ldots, x_m$ by forward iterations on the given map and thus we can recover X. Furthermore, we can also compute $N = Z - X$. Thus we have recovered both the original chaotic signal and noise. This noise resistance property is provided by a nonzero value of q. The larger the q, the higher the resistance to noise, but at the same time the length of the signal has to be longer in order to determine the initial condition x_1 more accurately from the symbolic sequence. Reconstruction error for the chaotic signal and noise is not zero owing to the problem of finding the initial condition from the symbolic sequence with limited precision.

B. Cascading noise-resistant maps

Vaidya goes one step further and defines a cascade of such noise-resistant maps. To add another chaotic signal Y to X, Vaidya defines a similar noise-resistant map which maps $[0,q)$ onto itself. It is self-similar to T_{noiseres}. Thus, he defines a whole cascade of noise-resistant maps, all of which are self-similar to the original one. The domain of succeeding maps reduces exponentially. For further details, please refer to Ref. 13.

With these cascades of maps, one could now add a whole family of chaotic signals $\{X_1, X_2, \ldots, X_k\}$ to one noise N (on the channel) with magnitude dictated by the number of maps to yield the signal Z. The symbolic sequence invariance is maintained at each step of cascading. Thus, the symbolic sequence of Z is used to decode X_1 and N_1, where N_1 is the sum of $\{X_2, X_3, \ldots, X_k\}$ and N. The symbolic sequence of N_1 is the same as that of X_2, and hence X_2 can be decoded. This procedure is repeated until all the signals are losslessly recovered along with N. Vaidya successfully applies this method to multiplex 20 discrete chaotic signals, each of length 350. Empirical results show that the reconstruction error for X is of the order of 10^{-10}. For noise N, the reconstruction error is of the order of 0.001 [the signal range is [0,1)]. Please refer to Ref. 13 for detailed results and discussion.

C. Drawbacks of method 1

The drawbacks of method 1 are as follows.

(1) Given discrete chaotic signals from various 1D chaotic maps, one has to find the corresponding signals in the noise-resistant binary/tent map using topological conjugacy which is not easy (and may not be possible always).

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(2) The amount of noise that can be added reduces exponentially as the number of chaotic signals to be multiplexed increases linearly.
(3) The reconstruction error for X is not zero. This is due to the problem of finding the initial condition from an arbitrarily long symbolic sequence with limited precision. The reconstruction error for N is also not zero.
(4) In principle it is possible to extend the idea of noise-resistant maps to other chaotic maps such as the logistic map. However, for each new map, the equations have to be worked out explicitly.

We are motivated to invent new methods of multiplexing which will circumvent the above problems. While exploiting the idea of symbolic sequence invariance under the addition of noise, we would like to devise a method which will work for any 1D chaotic unimodal map (and generalizable to other kinds of maps and higher dimensional ones) without the necessity of topological conjugacy. The scenario where the magnitude of noise is equal to that of the signal also needs to be addressed.

IV. METHOD 2

The key idea of method 1 is the notion of symbolic sequence invariance. As long as we ensure that the symbolic sequence of the original chaotic signal X is unaffected by adding noise N (uniformly distributed), the resulting signal $Z=X+N$ has the same symbolic sequence as $X(S(Z)=S(X))$. Then, given this arbitrarily long symbolic sequence $S(X)$, the problem reduces to determining its initial condition and iterating this initial condition to obtain the entire chaotic signal X. Once X is determined, one could subtract X from Z to obtain noise N.

In order to find the initial condition from an arbitrarily long symbolic sequence of a chaotic map, we make use of Algorithm I (see Appendix). Our implementation is for the skew-tent and skew-binary maps and can be extended to other maps. The standard tent map and binary map are part of this family. In a separate application,[14] Algorithm I is used for optimal lossless data compression for binary stochastic independent and identically distributed (i.i.d.) sources.

Method 2 is described as follows.

(1) Let $X_1, X_2, \ldots, X_k$ be k chaotic signals of length m to be multiplexed. Each of these signals is obtained from distinct initial conditions (randomly chosen) on the skew binary map ($p=0.499$). The skew binary map is given by the equations

$$
\begin{aligned}
x &\mapsto x/p, \quad x \in [0,p), \\
&\mapsto (x-p)/(1-p), \quad x \in [p,1).
\end{aligned}
$$

Here $0<p<1$. For $p=0.5$, it reduces to the standard binary map or Bernoulli shift map. However, we have chosen $p=0.499$, since for $p=0.5$ the map is a Shift map and with finite precision (~ 33 bits for initial condition), the chaotic trajectory will hit zeros after 33 iterations. With $p=0.499$, this is avoided.

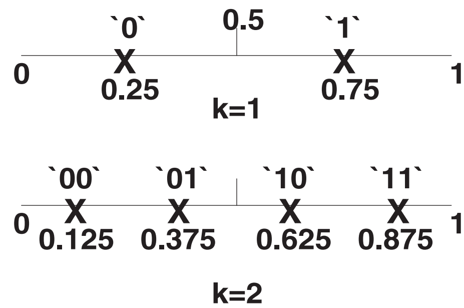

FIG. 3. Method 2: for cases $k=1$ and $k=2$. The points marked X are transmitted depending on the symbolic sequences of the chaotic signals. Owing to noise at the channel, the received signal will also be uniformly distributed in the range [0,1). This explains Fig. 5(c).

(2) Compute the symbolic sequence $\{SX_i\}_{i=1}^{i=k} = \{S(X_i(1)), S(X_i(2)), \ldots, S(X_i(m))\}_{i=1}^{i=k}$. The function $S(\cdot)$ is defined as follows:

$$
\begin{aligned}
S(x) &= 0, \quad \text{if} \quad 0 \le x < p \\
&= 1, \quad \text{if} \quad p \le x < 1.
\end{aligned} \tag{2}
$$

(3) Compute

$$
\begin{aligned}
\{S_i\}_{i=1}^{i=m} = \{&\langle SX_1(1)SX_2(1), \ldots, SX_k(1)\rangle_2, \\
&\langle SX_1(2)SX_2(2), \ldots, SX_k(2)\rangle_2 \cdots \\
&\langle SX_1(m)SX_2(m), \ldots, SX_k(m)\rangle_2\}.
\end{aligned}
$$

Here, $\langle\cdot\rangle_B$ denotes a number that is base-B representation.

(4) Compute $D=\{D_i\}_{i=1}^{i=m}$, where $D_i=\langle S_i\rangle_{10}$.
(5) Compute $Z=\{Z_i\}_{i=1}^{i=m}$, where $Z_i=(2D_i+1)/(2^{k+1})$.
(6) Transmit Z across the channel.
(7) Receive $Z_{\text{noisy}}=Z+N$. Here $N=\{N_i\}_{i=1}^{i=m}$, where each N_i is a uniformly distributed noise in the range $(-2^{-(k+1)}, +2^{-(k+1)})$.
(8) At receiver, compute $D_{\text{noisy}}=\lfloor 2^k Z_{\text{noisy}}\rfloor$, where $\lfloor(\cdot)\rfloor$ is the *floor* operation which computes the maximum integer that is less than the argument. Note that $D_{\text{noisy}}=D$. This is because $D<2^k Z_{\text{noisy}}<(D+1)$. The floor operator makes this equal to D since D is always a positive integer. This is where we have made use of the fact that the symbolic sequence is invariant in spite of noise. Here D has the information on the symbolic sequence of all the k chaotic signals.
(9) Once we have $D_{\text{noisy}}=D$, we can recover the symbolic sequences of each of the k chaotic signals and thereby recover the k initial conditions $\{X_1(1), X_2(1), \ldots, X_k(1)\}$ by Algorithm I (see Appendix).
(10) From the initial conditions, the k chaotic signals can be recovered by forward iterations.
(11) Noise can be recovered at the receiver by computing Z from D and computing $N=Z_{\text{noisy}}-Z$.

A. Experimental simulations

As a simple example, Fig. 3 shows the points transmitted for cases $k=1$ and $k=2$. Method 2 was experimentally simulated for $k=20$ chaotic signals of length $m=200$ each. They

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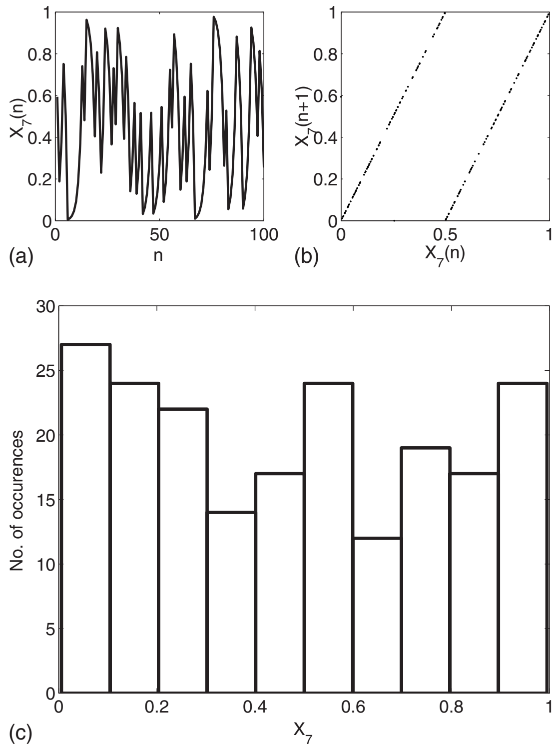

FIG. 4. Method 2: multiplexing of $k=20$ chaotic signals in presence of noise N. (a) the seventh chaotic signal $X_7(\cdot)$. $m=200$, but only first 100 are shown. (b) phase portrait of $X_7(\cdot)$. (c) histogram of $X_7(\cdot)$. Note: the initial conditions for the 20 chaotic signals are chosen with a precision of ten decimal digits ($\sim$33 bits). All subsequent iterates are in double precision. Noise is also in double precision.

were all generated by randomly chosen initial conditions on the skew binary map with $p=0.499$. The initial conditions for the 20 chaotic signals are chosen with a precision of ten decimal digits ($\sim$33 bits). All subsequent iterates are in double precision. Noise is also in double precision.

Figure 4 shows the seventh chaotic signal X_7. The phase portrait and the histogram are also shown. Figure 5 shows Z, the phase portrait of noise N, phase portrait of Z_{noisy} after the addition of noise, and the histogram of Z_{noisy}. Owing to white

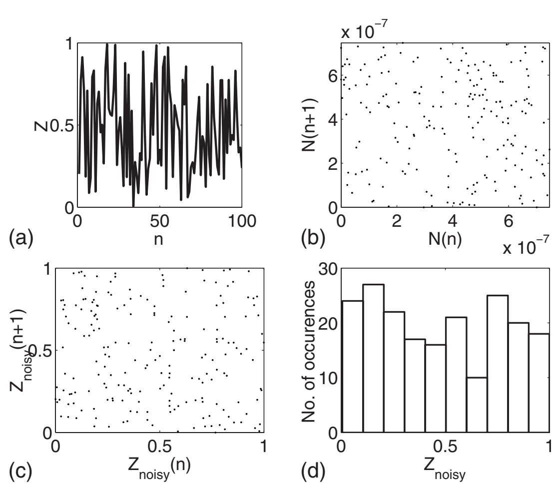

FIG. 5. Method 2: multiplexing of chaotic signals in presence of noise. (a) Z signal (showing first 100 values only). (b) phase portrait of noise N. (c) phase portrait of Z_{noisy}. (d) histogram of Z_{noisy}.

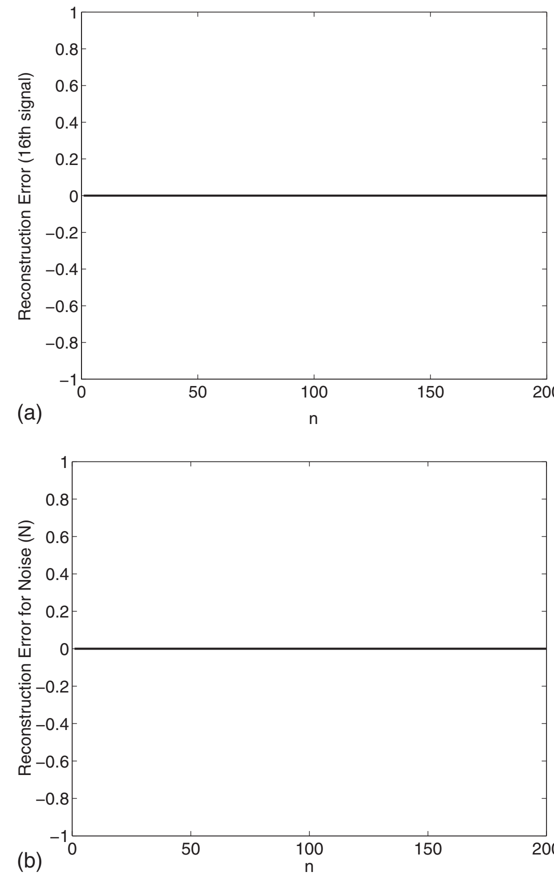

FIG. 6. Method 2: (a) reconstruction error for the 16th chaotic signal is zero. The reconstruction error is zero for all chaotic signals. (b) the reconstruction error for noise N is also zero.

noise, Z_{noisy} gives away no information about the number of signals or the type of signals being added. This is very conducive for secure communications. All the 20 chaotic signals and noise were successfully recovered in a lossless fashion at the receiver (see Fig. 6). This confirms the efficacy of method 2. This method was found to work with the same accuracy for single precision measurements.

Method 2 works in presence of a lossy channel, the noise being additive, but the magnitude of noise that is tolerated depends on the number of signals being multiplexed. As the number of signals (k) increases, the magnitude of noise (2^{-k}) that can be tolerated at the channel goes down exponentially. For $k=2$, this would imply robustness to 50% of noise (compare this with Liu and Davis' dual synchronization which is sensitive to even 1% of noise).

V. METHOD 3

The biggest advantage of method 2 is that in principle it works for any dynamical system. As long as we know the Markov partitions of the dynamical system, we can define the symbolic sequence and hence use method 2. There is no need of using topological conjugacy or construction of special noise-resistant maps like in method 1. However, one needs to develop an analog of Algorithm I (i.e., finding an initial condition corresponding to an arbitrary long symbolic sequence on the dynamical system) for the method to work.

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In method 3, the noise magnitude can be equal to the signal magnitude. However, we can no longer operate in scenario 1. We assume that we have control on the way in which noise is added (noise is still assumed to be uniformly distributed) and that the channel is lossless (scenario 2).

Method 3 is described as follows.

(1) Let $X_1, X_2, \ldots, X_k$ be k chaotic signals of length m to be multiplexed. Each of these signals is obtained from distinct initial conditions on the skew binary map ($p=0.499$ for the same reason as in method 2).

(2) Let noise be $N=\{N_i\}_{i=1}^{i=m}$, where each N_i is i.i.d. (uniform) in the range (0,1). Noise N is independently generated but available at the receiver.

(3) Given the two signals $A=\{A(i)\}_{i=1}^{i=m}$ and $B=\{B(i)\}_{i=1}^{i=m}$, where A is a chaotic signal (B can be anything), we define the operation $A+B\cdot S(A)=\{A(i)+B(i)\cdot S(A(i))\}_{i=1}^{i=m}$ as follows:

$$\begin{aligned} A(i)+B(i)\cdot S(A(i)) &= A(i)-B(i) \quad \text{if } S(A(i))=0 \\ &= A(i)+B(i) \quad \text{if } S(A(i))=1, \end{aligned}$$

where $S(\cdot)$ is defined in Eq. (2).

(4) Compute the following signals:

$$Z_1=\frac{X_1+N\cdot S(X_1)+1}{3}, \quad Z_2=\frac{X_2+Z_1\cdot S(X_2)+1}{3},$$

$$Z_3=\frac{X_3+Z_2\cdot S(X_3)+1}{3},$$

$$\vdots$$

$$Z_k=\frac{X_k+Z_{k-1}\cdot S(X_k)+1}{3}, \quad Z=Z_k.$$

(5) Transmit Z on the lossless channel. Note that the dynamic range of Z is [0,1).

(6) Receiver receives Z. By symbolic sequence invariance, we have the following identities:

$$S(Z)=S(Z_k)=S(X_k), \quad S(Z_{k-1})=S(X_{k-1}),$$

$$\vdots$$

$$S(Z_1)=S(X_1).$$

(7) We start with Z and compute $S(Z_k)$. By the first identity, we have $S(X_k)$. Algorithm I (see Appendix) is applied to determine $X_k(1)$. Hence X_k is recovered losslessly. Knowing X_k and $S(X_k)$, we can compute Z_{k-1} by the following equation:

$$\begin{aligned} Z_{k-1}(i) &= X_k(i)-3Z(i)+1 \quad \text{if } S(X_k(i))=0 \\ &= 3Z(i)-X_k(i)-1 \quad \text{if } S(X_k(i))=1. \end{aligned}$$

(8) Knowing Z_{k-1}, we repeat the procedure to extract X_{k-1} and Z_{k-2}. This is repeated until we have extracted all the chaotic signals and noise N. Note that noise can be thought of as Z_0 and the same procedure applies.

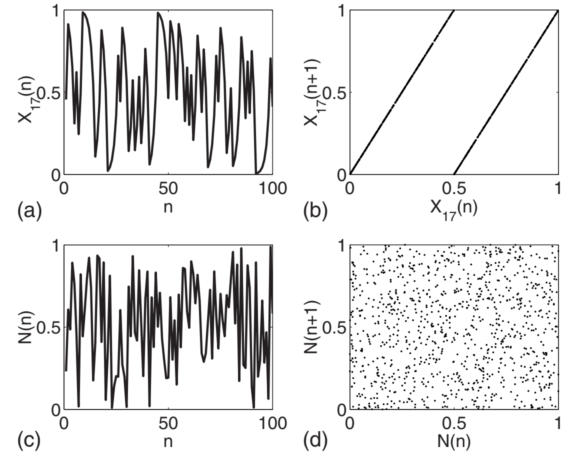

FIG. 7. Method 3: multiplexing of $k=24$ chaotic signals in presence of noise. (a) the 17th chaotic signal $X_{17}(\cdot)$. The length $m=1000$, but only first 100 are shown. (b) phase portrait of $X_{17}(\cdot)$. (c) noise $N(\cdot)$. (d) phase portrait of $N(\cdot)$. Note that magnitude of N is the same as that of X. The initial conditions for the 24 chaotic signals are chosen with a precision of ten decimal digits (~33 bits). All subsequent iterates are in double precision. Noise is also in double precision.

A. Experimental simulations

Method 3 was experimentally simulated for $k=24$ chaotic signals of length $m=1000$ each and noise N of the same length. The chaotic signals were all generated by randomly chosen initial conditions on the skew binary map ($p=0.499$). Figure 7(a) shows the 17th chaotic signal X_{17}. The phase portrait is shown in Fig. 7(b). Figure 7(c) shows noise N which has the same magnitude as that of the chaotic signal. The phase portrait of noise N is shown in Fig. 7(d). The final signal that is transmitted on the lossless channel Z is shown in Fig. 8(a). Its phase portrait and histogram are shown in Figs. 8(b) and 8(c), respectively. Note that in this method, the final signal Z does not have a uniform distribution although the individual chaotic signals and noise are uniform. This is because of the special way in which the signals were added.

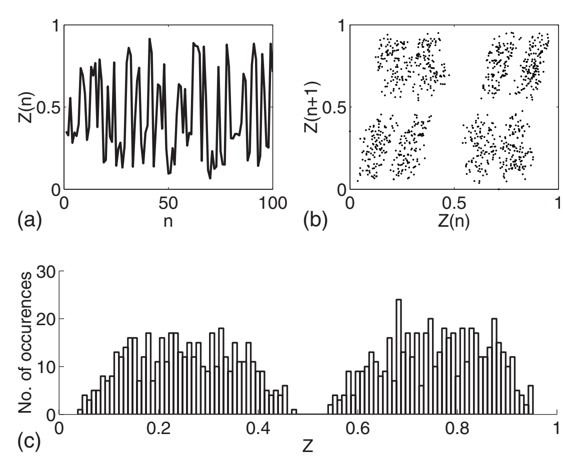

FIG. 8. Method 3: multiplexing of chaotic signals in presence of noise. (a) Z signal generated by method 3 is not uniformly distributed. This is transmitted on the lossless channel. (b) phase portrait of Z. (c) histogram of Z showing that it is bimodal.

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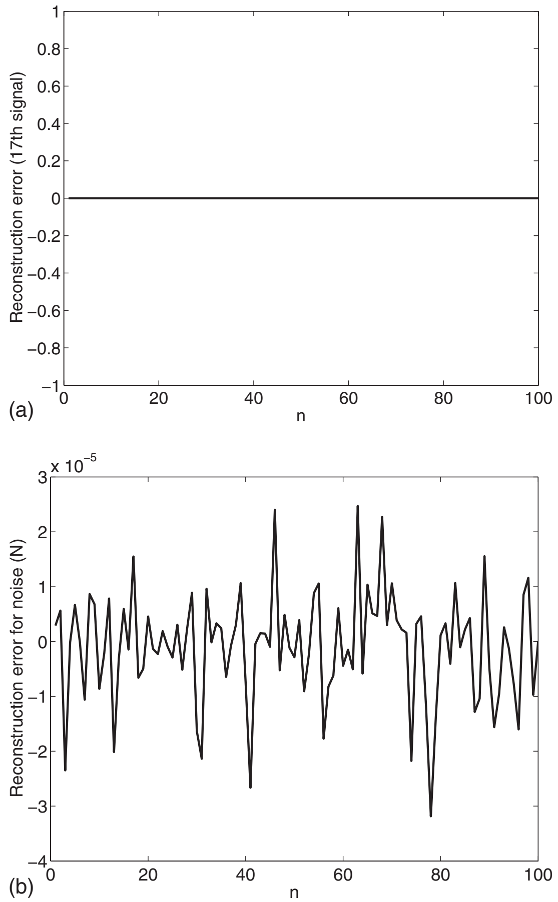

FIG. 9. Method 3: (a) reconstruction error for the 17th chaotic signal is zero. The reconstruction error is zero for all chaotic signals. (b) the reconstruction error for noise N is not zero. This is because of division by 3 at each step which causes error.

All the 24 chaotic signals were successfully recovered in a lossless fashion at the receiver. As an example, Fig. 9(a) shows that the reconstruction error for X_{17} is zero. The reconstruction error for noise N is not zero, as indicated in Fig. 9(b). Division by 3 in step (4) results in roundoff errors because of finite precision (all calculations are performed in double precision). This cannot be recovered in step (7). Even with single precision measurements, the reconstruction error for all chaotic signals is zero. This confirms the efficacy of method 3.

VI. REMARKS ON THE THREE METHODS

The following observations can be made on the three methods.

(1) The idea of symbolic sequence invariance is the key to the success of all the three methods. The way this idea is implemented is different in the three methods.
(2) The way noise is handled is the same in methods 1 and 2 since there is no control on noise in scenario 1. Scenario 2 is much more restrictive in terms of noise.
(3) All the three methods rely on Algorithm I—finite precision implementation of finding the initial condition given an arbitrarily long symbolic sequence of the chaotic signal. The method of extracting the symbolic sequence from a "noisy nonlinear system" and applying Algorithm I is analogous to filtering out noise in linear systems by means of integration or other linear filters (for, e.g., low pass filters).
(4) Methods 2 and 3 can be easily extended to tent map, skew-tent map, logistic map, and other unimodal maps. It is also possible to extend the methods to nonunimodal 1D maps and possibly to higher dimensional maps. The key is to find an analog of Algorithm I in those cases.
(5) Method 1 has a hint of using the idea of "forbidden symbol"[15] by allocating the length q on the interval which is never used by the map. This method can be potentially used for error detection[16] and correction.
(6) In method 2, just by observing the signal on the channel, no information can be gleaned. The distribution is uniform and the phase portrait is also random looking. The fact that multiple chaotic signals have been embedded is not obvious. This property enables it to be used in steganography or information hiding. In least significant bit (LSB) steganography, the LSB of natural signals is replaced by the secret (noise or noiselike). Method 2 is doing the reverse: most significant bit (MSB) steganography, where the MSB of the secret (noise or noiselike) is replaced by the symbolic sequence of the chaotic signal.
(7) Method 3 can handle any number of chaotic signals and noise. However, in practice there will be limitation on the number of signals owing to finite precision since we are rescaling the range of the signals to [0,1) (by addition of 1 and division by 3). The reconstruction error for noise will increase with increasing number of signals multiplexed.
(8) The methods show that chaotic signals are highly redundant and hence robust to noise. As long as the symbolic sequence is preserved, the actual signal can be distorted to a great deal. Also, forward iteration of chaotic dynamical systems exhibits sensitive dependence on initial conditions, but backward iteration shows resistance to noise. These features are not exhibited by random/stochastic signals or periodic signals. This favors the use of chaotic signals for multiuser communication applications. This also makes a strong case for why biological systems may use chaotic signals for transmission of information. Neuronal signals may use similar mechanisms for robust transport of information.
(9) The above methods will not work for purely random signals or for nonchaotic signals since there is no way to reconstruct the entire trajectory by knowing the symbolic sequence. The redundancy of chaotic signals is necessary. At the same time, chaotic signals appear "random" in distribution (for, e.g., skew-binary and skew-tent maps have uniform distribution as the invariant distribution, hence they are used in chaotic cryptography[17]).
(10) In contrast with earlier works on multiplexing discrete chaotic signals using chaotic synchronization, our methods can multiplex more than two discrete chaotic signals (although Tsimring[4] can multiplex multiple discrete chaotic signals, we listed the drawbacks ear-

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lier). There are no coupling coefficients used in our methods and hence there is no condition to be satisfied for multiplexing. Unlike Tsimring's method, we only need to transmit a single scalar signal through the channel (like conventional multiplexing methods). As noted previously, earlier methods based on chaotic synchronization are vulnerable to noise. Methods 1 and 2 can tolerate noise but limited by the number of signals that is added. However, in method 3, the noise magnitude is the same as that of the signals, but the reconstruction error for noise is not zero (reconstruction error for the chaotic signals is zero).

(11) An important point to note is that both of our methods have been found to work even if single precision is used for all the iterates of the chaotic signals. Our methods require less iterates than those typically required by chaotic synchronization. Tsimring's method[4] requires 10 000 iterates whereas both methods 2 and 3 proposed in this paper work with as less as 35 iterates. This is true irrespective of the number of chaotic signals multiplexed. Liu and Davis[5] require different number of iterates for different coupling constants. The best they achieve for multiplexing two cosine maps is less than ten iterates and for two logistic maps it is around 50 iterates. But, these numbers are for multiplexing only two pairs of maps.

(12) The algorithmic complexity of our methods is quite low since the operations that are involved are comparisons (for finding the symbolic sequence) and simple operations (addition, multiplication by 3, etc.).

(13) There is no violation of Shannon's theorems for information transmission in any of the methods. By transmitting the entire trajectory, we are sending lots of bits, much more than actually required for sending only the initial condition. These methods are not meant for compression of data. These are mechanisms to exploit the inherent redundancy in chaotic signals in spite of noise.

VII. CONCLUSIONS AND OPEN PROBLEMS

In this work, we have proposed new methods for multiplexing of discrete chaotic signals in presence of noise without employing chaotic synchronization. By using the idea of symbolic sequence invariance, we were able to "add" several chaotic signals and "separate" them losslessly at the receiver. We can either have a lossy channel but with limited noise (methods 1 and 2) or have a lossless channel with noise having the same magnitude as the signal added in a very special way to the sender (method 3). These are suitable for multiuser communications and steganographic applications. An open problem is to investigate whether one can have both features in a single method.

The inherent redundancy and structure in chaotic signals which otherwise appear random in probability distribution can be harnessed for robust communication of information. It is quite likely that such efficient mechanisms (or similar ones) of handling noise in dynamical systems are already being employed in naturally occurring physical and biological systems.

Compared to some of the existing methods for multiplexing discrete chaotic signals based on chaotic synchronization, our methods offer several advantages such as zero reconstruction error for the chaotic signals even with limited precision measurements and lesser number of iterates. The newly proposed methods can handle multiple signals from multiple maps (including Bernoulli shift or the binary map which was not possible by the method of Liu and Davis) and offer good noise resistance capability. An electronic circuit implementation based on these methods needs to be investigated.

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APPENDIX: FINITE PRECISION IMPLEMENTATION OF FINDING THE INITIAL CONDITION ON A DISCRETE CHAOTIC MAP FROM AN ARBITRARILY LONG SYMBOLIC SEQUENCE

Given a very long (but finite) symbolic sequence on a known discrete chaotic map, the initial condition is determined by a backward iteration. This actually results in an interval (since the symbolic sequence is finite in length) and the midpoint of the interval can be used as the initial condition to recreate the symbolic sequence. As the length of the symbolic sequence increases, the interval in which the initial condition is going to lie shrinks in size. This creates a problem in performing the backward iteration on a finite precision computer as the two ends of the interval come very close to each other and at some point it would be no longer possible to continue with the backward iteration. This problem needs to be addressed by some kind of renormalization or rescaling of the interval in order for the method to be useful for finding an initial condition for very long symbolic sequences.

The idea is as follows. As soon as the interval completely lies to the left of 0.5, the final initial condition will have a "0" in its binary expansion ("1" if the interval is completely to the right of 0.5). Hence, this can be written as output and the current interval can be doubled in length. This ensures that the two ends of the interval will never come close to each other. At the end of all the iterations, the midpoint of the final interval is written as output. The only case in which this method would fail is when the interval straddles the point 0.5 at every iteration. The probability of this happening exponentially decreases with each iteration. To handle this special case, there can be a check on the size of the interval and once it reduces to a certain value, we stop reading in more symbols and force an output. The interval is reset to $[0,1)$. The iteration starts afresh for the next incoming bits. This would increase the number of output bits slightly as we are not encoding the entire symbolic sequence to determine a single initial condition. This increase in number of bits is negligible for long sequences.

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ALGORITHM I

1. Input: Arbitrarily long finite symbolic sequence M of length N on the skew binary map with known parameter p.
2. Create Markov partitions—$[0,p)$ corresponding to symbol ‘0’ and partition $[p,1)$ corresponding to symbol ‘1’.
3. Initialize interval $[L,U)$ to [0,1). Initialize $Tol=10^{-8}$.
4. Initialize $k=1$.
5. Input the kth bit from M.
6. If the bit is 0, then set:
   $L \leftarrow L/p, \quad U \leftarrow U/p$.
   else, set:
   $L \leftarrow (L-p)/(1-p), \quad U \leftarrow (U-p)/(1-p)$.
   Set $k \leftarrow k+1$.
7. If $0 \leq L, U < 0.5$, then Output bit ‘0’ and set:
   $L \leftarrow 2L, \quad U \leftarrow 2U$.
   if $0.5 \leq L, U < 1$, then Output bit ‘1’ and set:
   $L \leftarrow 2L-1, \quad U \leftarrow 2U-1$.
8. If $(U-L) \leq Tol$, Output $x_{ic}=(L+U)/2$ in binary representation and reset $[L,U)=[0,1)$.
9. If $k \leq N$, go to step 5, else continue to step 10.
10. If $L \leq 0.5 \leq U$, then $x_{ic}=0.5$ else $x_{ic}=(L+U)/2$.
11. Output x_{ic} in binary representation.

For multiplexing of discrete chaotic signals proposed in this paper, we chose $p=0.499$ since we are using the skew binary map. The algorithm described here can be extended easily to find the initial condition from an arbitrarily long symbolic sequence of other chaotic maps (for, e.g., to find the initial condition on the skew-tent map, equation in step 7 needs to be modified appropriately). A similar extension can be done for the logistic map as well. The algorithm can also be extended to handle more than two Markov partitions (the symbolic sequence is no longer binary).

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[18]Steganography is the art and science of hiding secret data in a message. Unlike cryptography, where it is known that the data are encrypted and meant to be secret, in steganography, the very presence of the secret information is hidden.

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