Polygon Inequality-Based Genuine Entanglement Measures

Sumit Chilkoti[1] **and Mrittunjoy Guha Majumdar**[1,2,3*]

[1]Department of Physics, Amrita Vishwa Vidyapeetham, Kollam, India
[2]School of Artifical Intelligence, Amrita Vishwa Vidyapeetham, Faridabad, India
[3]School of Natural Sciences and Engineering, National Institute of Advanced Studies, Bengaluru, India
[*]e-mail: mrittunjoy@dl.amrita.edu

ABSTRACT

Genuine multipartite entanglement (GME) represents the strongest form of quantum correlation, where all parties of a system across each bipartition are inseparably linked. This resource is crucial for advanced quantum protocols, including multi-party cryptography and distributed quantum computing, yet its quantification remains a significant theoretical challenge. This review surveys a rapidly evolving geometric paradigm that translates this challenge into a problem of polygon inequalities. By representing marginal bipartite entanglements as side lengths, the distribution of entanglement in a multipartite state must satisfy geometric closure conditions akin to forming a polygon. These polygon-based constraints provide a unifying framework for detecting and quantifying GME, offering an intuitive bridge between bipartite entanglement measures and global multipartite structure. We chart the development of this approach from its monogamy-based origins to modern formulations using concurrence triangles, wedge products, and entropy-based polygons. Recent advances have established rigorous conditions for these geometric quantities to serve as faithful, operationally meaningful entanglement measures, while also extending the framework to mixed states and higher-dimensional systems. Despite progress, key challenges persist in ensuring universal monotonicity and experimental robustness. By synthesizing these geometric insights, this review highlights how polygon inequalities are reshaping the quantification of complex quantum correlations and providing novel diagnostic tools for the next generation of quantum technologies.

Key points:

- The genuine multipartite entanglement is quantified using polygon inequalities, where the closure condition of a polygon, inspired by the CKW framework, signals the global inseparability of the state.
- Extensive studies in the last decade have evolved the polygon inequality framework into rigorous, axiom-satisfying measures of entanglement, providing an intuitive geometric framework for resource theory.
- The versatility of the polygon inequality paradigm has been successful in generalising the measures beyond concurrence to various entanglement monotones, higher-dimensional systems, and mixed states.
- Despite progress, key challenges remain in ensuring universal monotonicity under all quantum operations, developing efficient experimental certification protocols, and fully integrating the framework with operational quantum tasks.

Introduction

Entanglement is the cornerstone of several quantum information and computing protocols, such as superdense coding[1], teleportation[2], and quantum computing[3]. Entanglement is a feature of quantum systems that violates local realism, thereby surpassing classical correlation bounds[4]. However, quantum information theory treats entanglement as a ‘quantifiable resource’, enabling its extensive application across various tasks related to quantum information processing[5]. Therefore, entanglement is treated within the framework of a quantum resource theory under LOCC (local operation and classical communication), necessitating the development of operationally meaningful methods of its quantification[6]. Since the mid-1990s, substantial effort has been devoted to formalising such quantification through entanglement measures. An entanglement measure E is a functional that maps the density operator $(\rho_{AB}) \in B(H_{AB})$ to a non-negative real number, typically normalized such that $E(\rho_{AB}) \in [0,1]$[7,8]. The early attempts to quantify entanglement employed operationally defined resource measures: entanglement of formation E_F and distillable entanglement E_D[9]. Evaluating E_D and E_F for arbitrary states is computationally demanding, as it requires optimization over an asymptotically large number of state copies. Consequently, attention shifted toward single-copy entanglement measures, which are more amenable to analytical and numerical treatment. Following fundamental developments, several bipartite entanglement measures have been designed, such as robustness of entanglement[10], negativity[11], logarithmic negativity[12], and entropy of entanglement[13]. Multiple seminal studies have adopted an axiomatic approach to delineate criteria for accurate entanglement measures, such as monotonicity under LOCC, nullity for separable states, convexity, and additivity for independent subsystems, which support their practical and theoretical importance in the field of quantum information science[6,8,14]. Concurrence was one of the initial computationally tractable bipartite entanglement measures, presenting a closed-form expression for the entanglement of formation[15,16]. Subsequently, refined and generalised definitions of concurrence have developed across numerous studies[17–20].

Quantifying genuine multipartite entanglement (GME) is important, as it holds a distinguished role over bipartite or general multipartite entanglement in quantum information science owing to its unique ability to enable quantum protocols that demand collective coherence and security among multiple parties, such as multi-party quantum teleportation[21], quantum secret sharing[22], and quantum key distribution[23], which bipartite entanglement cannot accomplish effectively. The complete classification and knowledge of intricate multipartite entanglement structure still remains an open problem[6,24–27]. GME differs fundamentally from general multipartite entanglement because GME measures must vanish for all biseparable states while remaining strictly positive only for states exhibiting entanglement across every bipartitions[28,29]. Unlike conventional bipartite entanglement measures, GME measures need to account for the richer structure of multipartite systems, therefore distinguishing fully multipartite entanglement from all biseparable mixtures and reflecting the strictly stronger correlation structure present in multipartite systems[28]. Consequently, GME measures often generalize bipartite concepts, such as concurrence and negativity, but must satisfy stricter conditions like monotonicity under local operations and classical communication (LOCC) for all bi-partitions, invariance under local unitary transformations, and convexity[30,31]. This refined distinction from bipartite entanglement complicates the formulation of GME measures, requiring innovative mathematical frameworks that go beyond simple generalisations of bipartite entanglement measures.

Geometry has been a novel angle that has helped in characterizing entanglement at a deeper level[32–35]. Geometric approaches to quantifying entanglement have defined measures based on distance and angles from separable states[34,36–40]. Motivated by the role of geometry in quantifying entanglement, this review examines a unique theoretical approach to quantify multipartite entanglement that is built upon a novel approach which utilises marginal bipartite entanglements across bi-partitions[41,42]. Extending bipartite entanglement measures to the multipartite domain reveals that different bi-partitions encode complementary forms of correlation; for instance, a tripartite state ρ_{123} admits six bipartite entanglements, including one-to-group terms such as $E_{i|jk}(\rho_{ijk})$ and one-to-one terms like $E_{i|j}(\rho_{ij})$. Following the framework of the (CKW) inequality[43], the magnitudes of 'one to group' and 'one to one' entanglements are interrelated through monogamy relations that constrain the distribution of entanglement across subsystems. By exploiting these constraints, one can formulate polygon-based inequalities[33,41,44] that serve as geometric indicators or candidates for GME[27,45–47]. Monogamy relations between marginal entanglement proved pivotal for generalising several bipartite measures to multiparty states. Among these, concurrence has been one of the most extensively employed and is also among the first bipartite measures to be generalised to higher-dimensional systems[17,48,49]. This framework provides an intuitive bridge between bipartite measures and global multipartite structure, establishing a coherent geometric foundation for the study of GME.

This review examines a geometric approach to genuine multipartite entanglement (GME) quantification based on polygon inequalities among marginal bipartite correlations. We first survey the current landscape of GME characterization—including definitions via biseparability, k-separability hierarchies, and existing measure constructions. Then we introduce the novel framework which is geometric centric, but departs from the conventional ideas of geometrical GME measures. We classify the framework into various approaches utilised in various studies and do a thorough comparison. We then discuss experimental and application based studies where this framework is utilised. We end with certain open questions and a future outlook. This review aims to provide a comprehensive and critical synthesis of this rapidly evolving geometric paradigm. We trace its inspiration from early monogamy-based insights to its current formulation as a rigorous framework for GME quantification. Our central thesis is based on the polygon inequality as an intuitive bridge between experimentally accessible bipartite data and the abstract structure of global entanglement, while simultaneously providing a fertile mathematical foundation for addressing long-standing open problems in quantum resource theory. By critically analysing the advantages and limitations of various polygon-based measures—from concurrence triangles to wedge product hypervolumes—we will demonstrate how this geometric perspective is inhancing our understanding of multipartite correlations and paving the way for practical certification protocols in emerging quantum technologies. For more information, we direct the reader to the references[50,51].

Genuine Multipartite Entanglement

Genuine multipartite entanglement (GME) captures the strongest form of non-classical correlations in composite quantum systems, corresponding to states in which all parties are entangled in an intrinsically inseparable way. While bipartite entanglement is now well understood in terms of canonical forms, measures, and operational tasks, the multipartite setting admits a much richer structure of entanglement types and separability classes, for which no single canonical characterization exists[52,53]. In this landscape, GME is distinguished from more general multipartite entanglement by the requirement that the state cannot be written as a mixture of states that are separable across any bipartition, making it the natural resource underlying genuinely global quantum correlations and many-body protocols that cannot be simulated by entanglement confined to subsystems[54,55]. Even for three qubits, the structure of multipartite entanglement departs dramatically from the bipartite paradigm. The seminal study by Dür, Vidal, and Cirac[56] showed that there exist at least two inequivalent classes of genuine three-qubit entanglement under stochastic local operations and classical communication (SLOCC), exemplified by the GHZ and W states, which cannot be converted into one another even probabilistically by local means[57]. This inequivalence indicates that not all genuinely multipartite entangled states are of the same 'kind', and that GME cannot be captured by a single scalar monotone in a way analogous to the Schmidt rank in the bipartite case. The classification problem only becomes more intricate as the number of parties grows, with a proliferation of SLOCC classes and intermediate forms of partial separability that complicate both the conceptual and quantitative treatment of multipartite entanglement.

In response to this complexity, a number of approaches have been developed to define and quantify GME in a principled way. The first was the introduction of the genuine multipartite concurrence, which had associated axioms that a GME measure should satisfy—vanishing on all biseparable states, being strictly positive on genuinely entangled states, invariant under local unitaries, monotonic under LOCC and being convex under mixing—and provided computable lower bounds based on bipartite concurrences of reduced density matrices[58]. A systematic framework for multipartite entanglement measures built on the lattice of separability classes was subsequently proposed, in which GME appears as the topmost layer in a hierarchy that also includes various forms of k-separability and k-producibility[52,59,60]. These works collectively emphasize that meaningful GME quantifiers must be sensitive to the global inseparability of the state rather than merely aggregating bipartite correlations. More recently, it has been shown that GME is not only a static structural notion but also an operational resource that can be activated in nontrivial ways[53]. In particular, the work of Yamasaki[53] demonstrates that states which are not genuinely multipartite entangled in a single-copy setting can nevertheless yield GME when multiple copies are available, revealing a non-additive behaviour in the resource theory of GME that has no analogue in the simplest bipartite scenarios. Together with the foundational perspective provided by Horodecki[6], which emphasised the central role of GME in quantum networks, nonlocality, and many-body physics, these developments position genuine multipartite entanglement as a central organising concept for understanding multipartite quantum correlations. The following subsections build on this foundation by contrasting GME with generic multipartite entanglement, setting out formal requirements for GME measures, and situating GME within the broader hierarchy of k-separable and k-producible states[61].

General Multipartite Entanglement versus Genuine Multipartite Entanglement

In bipartite systems, any given state is either separable or entangled, and this distinction is exhaustive. In multipartite systems, by contrast, there is a whole spectrum of intermediate structures between full separability and the strongest form of global entanglement. A state of N parties is called *fully separable* if it can be written as a convex mixture of product states over all single-party subsystems; it is called *multipartite entangled* if it is not fully separable. However, multipartite entanglement in this broad sense still allows for the possibility that entanglement is confined to proper subsets of parties or that the state is a mixture of differently factorized components[54]. Formally, an N-partite pure state $|\psi\rangle$ is called *biseparable* if there exists a bipartition $A|\bar{A}$ of the parties such that

$$|\psi\rangle = |\phi_A\rangle \otimes |\phi_{\bar{A}}\rangle\,. \tag{1}$$

A mixed state ρ is biseparable if it admits a decomposition

$$\rho = \sum_i p_i\, |\psi_i\rangle\langle\psi_i| \tag{2}$$

in which every pure component $|\psi_i\rangle$ is biseparable, possibly with respect to different bipartitions[54,58]. If a state is not biseparable, it is called genuinely N-partite entangled (GME)[58,62]. This definition implies that the set of biseparable states is convex and strictly contains all fully separable and partially separable states; GME is precisely the complement of this convex set within the state space[26,30].

This distinction has both structural and operational significance. Structurally, biseparable states can be generated without ever requiring an interaction that couples all parties simultaneously: they can be prepared by entangling subsets in turn and then mixing, so they do not encode irreducible N-body correlations[54]. Genuinely multipartite entangled states, by contrast, cannot be simulated as mixtures of such partially entangled configurations and thus require genuinely global resources to create[58]. Operationally, many multipartite tasks—such as certain forms of quantum secret sharing, multi-party teleportation, or distributed protocols that rely on nonlocal correlations across every cut—derive their advantage specifically from GME rather than from generic multipartite entanglement[62, 63]. From the perspective of quantification and detection, this difference necessitates dedicated tools. Multipartite entanglement witnesses or measures that simply test "non-full separability" are insensitive to whether correlations are confined to few-body clusters or genuinely involve all parties. Genuine multipartite entanglement measures, in contrast, are explicitly required to vanish on all biseparable states and to be strictly positive only for GME states, and many modern criteria—ranging from concurrence-based constructions to positive-map criteria—are built to respect exactly this boundary[30, 58]. In what follows, this review adopts the GME notion as the primary target: polygon inequalities, geometric polytopes and monogamy-based constructions are analysed in terms of how effectively they distinguish genuine multipartite entanglement from the broader class of generic multipartite entangled states[62, 63].

k-Separability and Hierarchical Classifications

While genuine multipartite entanglement (GME) captures the strongest form of multipartite correlations, the full structure of entanglement in N-party systems reveals a rich hierarchy of partially separable states between full separability and GME[52, 59]. A mixed state ρ of N subsystems is k-separable if it can be written as a convex combination of states where each pure state factorizes into at most k non-trivial factors (clusters), i.e.,

$$\rho = \sum_i p_i \bigotimes_{j=1}^{m_i} |\psi_i^{(j)}\rangle\langle\psi_i^{(j)}|_{S_i^{(j)}}, \qquad m_i \geq k, \tag{3}$$

where $\{S_i^{(j)}\}_{j=1}^{m_i}$ is a partition of the N parties into m_i non-overlapping clusters, and each $|\psi_i^{(j)}\rangle$ is a state of the parties in $S^{(j)}$, with $m_i \leq k$ and $\sum_i p_i = 1$[52]. Equivalently, no partition of the N parties into more than k groups exists across which ρ is fully separable[59]. This notion induces a lattice-theoretic classification of partial separability: the set of k-separable states forms a convex cone, and the partial order $\rho_1 \preceq \rho_2$ if ρ_2 lies in the convex hull generated by states more separable than ρ_1 yields the down-set lattice of subsystem partitions[52, 59]. GME states occupy the top of this hierarchy as $(N+1)$-inseparable states, while k-producible states (convex combinations of pure states with at most k entangled parties) provide the dual perspective[60]. Szalay revealed a partial duality between k-separability and k-producibility, mediated by permutation-invariant properties labeled by Young diagrams under refinement order, with k-stretchability emerging as a balanced interpolant sensitive to both maximal cluster size and minimal uncorrelated subsystems[60].

Early classifications for qubits, such as normal forms for three- and four-qubit states under SLOCC, provided concrete examples but lacked generality for mixed states or higher dimensions[64]. A comprehensive review[26] formalised witnesses and criteria for biseparability and k-separability, and then later a hierarchical detection scheme was introduced using tailored entanglement witnesses that probe GME by ruling out all lower k-separable classes sequentially[30]. A Bloch-representation criterion complemented this by offering necessary conditions for $k$separability via positive maps on marginals[65]. These developments define a precise framework for polygon inequalities structured marginal-entanglement vectors for GME states must lie outside the feasible region defined by the k-separable polytopes for all $k < N$, providing a geometric diagnostic for hierarchy exploration[59, 60].

Requirements for Genuine Multipartite Entanglement Measures

The axioms to establish criteria for faithful GME quantifiers are formulated within the framework of Local Operations and Classical Communication (LOCC) such that operationally valid measures must stay monotonous; i.e, entanglement cannot increase under local manipulations[6]. The seminal classification distinguished the three-qubit pure states into four inequivalent classes under Stochastic LOCC (SLOCC): fully separable (product) states, biseparable states, GHZ-class states, and W-class states[56]. The latter two classes show GME, which means that they can't be separated by any bipartition. GHZ states show the strongest three-party correlations that disappear when a single qubit is lost, and W states show distributed entanglement that is robust against particle loss but doesn't show the strongest violation of tripartite Bell inequalities.[56].

Using this classification, four axiomatic requirements (GE1–GE4) for genuine multipartite entanglement measures were formulated[28, 29]:

1. *GE1 (Biseparable-zero)*: The measure $\mathscr{E}_{\text{GME}}(\rho)$ vanishes on all biseparable states, i.e., if $\rho = \sum_i p_i \rho_i^{J_i|\bar{J}_i}$ where each ρ_i is separable across some bipartition $J_i|\bar{J}_i$, then $\mathscr{E}_{\text{GME}}(\rho) = 0$[28].

2. *GE2 (Faithfulness)*: The measure should be strictly positive on all genuinely entangled states[50].

3. *GE3 (Convexity)*: For any ensemble $\{p_i, \rho_i\}$, the measure must satisfy $\mathscr{E}_{\mathrm{GME}}\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i \mathscr{E}_{\mathrm{GME}}(\rho_i)$, ensuring mixtures do not suddenly increase measured entanglement[6,28].

4. *GE4 (LOCC monotonicity)*: Under any LOCC protocol Λ, the measure should not increase on average: $\mathscr{E}_{\mathrm{GME}}(\Lambda[\rho]) \leq \mathscr{E}_{\mathrm{GME}}(\rho)$, adhering with the resource-theoretic framework[28].

These axioms distinguish GME measures from general multipartite entanglement measures, which may detect correlations across specific bipartitions without certifying entanglement across all partitions. For example, the residual tangle $\tau_{ABC} = C^2_{A|BC} - C^2_{AB} - C^2_{AC}$, satisfies LOCC monotonicity (GE4) but violates faithfulness (GE2) by vanishing identically on W-class states despite being a GME state[6]. In another instance, the geometric measure of entanglement, which is defined as the minimal squared Hilbert-Schmidt distance from separable states, satisfies faithfulness criteria but fails convexity (GE3), making it unsuitable as a resource monotone[37].

The stringency of these requirements reflects the fundamental challenge in multipartite scenarios: unlike bipartite entanglement where a single Schmidt decomposition fully characterizes pure-state structure, multipartite systems admit inequivalent SLOCC classes (e.g., GHZ vs. W) that cannot be interconverted even probabilistically[56]. A faithful GME measure must therefore detect entanglement across *all* possible bipartitions simultaneously, while remaining blind to the specific SLOCC class—a constraint satisfied by certain polygon-based constructions through symmetric treatment of all one-versus-rest marginals[29,66].

Additional desirable properties, though not strictly required, include:

- *Monogamy*[67]: GME measure must respect inequalities constraining entanglement shareability, e.g., $\mathscr{E}^{\alpha}_{A|BC} \geq \mathscr{E}^{\alpha}_{AB} + \mathscr{E}^{\alpha}_{AC}$ for appropriate α[68].

- *Additivity on tensor products*: $\mathscr{E}_{\mathrm{GME}}(\rho \otimes \sigma) = \mathscr{E}_{\mathrm{GME}}(\rho) + \mathscr{E}_{\mathrm{GME}}(\sigma)$ for independent systems, though recent activation phenomena demonstrate certain biseparable states violate this[69].

The interplay between these requirements motivates the polygon inequality framework: by constraining marginal bipartite entanglement measures across all partitions via geometric closure conditions (triangle inequalities, polytope facets), polygon-based approaches naturally enforce biseparable-zero conditions (GE1) while inheriting LOCC monotonicity (GE4) from underlying bipartite measures, provided appropriate power-law parameterization is employed[29,47].

Polygon-Based Approaches to GME Quantification

Geometric perspectives have played a significant role in the theoretical development of quantum entanglement. Be it distance-based entanglement quantifiers, hierarchies of entanglement classes, or eigenvalue spectra of reduced density matrices, the geometrical nature of entanglement has provided insight into the intrinsic structure of entanglement. With geometry as a central organising principle, a novel approach to quantum entanglement quantification is adopted in which the distribution of entanglement across one-to-group bipartitions is constrained by polygon-like inequalities that capture the global compatibility and sharing structure of multipartite correlations. The theory of bipartite entanglement measures was sufficiently developed, while multipartite entanglement remained an enigma. The resolution came from the seminal study of Coffman, Kundu and Wooters[43] where the term 'distributed entanglement' came into existence, owing to monogamy of entanglement. Though related, its important to highlight that the polygon-based approach is not based on monogamy of entanglement; the polygon inequalities provide the upper bound on the entanglement, whereas the CKW relations highlight the lower bound. Nevertheless, monogamy has emerged as a central theme, with both the regimes in which it is satisfied and those in which it is violated being actively explored. Rather than being the focus of this review, monogamy mainly serves to introduce the bipartition perspective that underlies multipartite entanglement. In subsequent sections we shall study the extensive development that led to the development of the polygon-based approach, which was first developed by[41].

CKW inequality-based monogamy relations were first applied to the tripartite state using squared concurrences of one to using squared concurrence to establish another multipartite measure, known as 'tangle'[43]. For a three-qubit state ρ, the CKW inequality is expressed as:

$$C^2_{1|23}(\rho) \geq C^2_{12}(\rho_{12}) + C^2_{13}(\rho_{13}) \tag{4}$$

This inequality motivates the presence of 'residual entanglement, which was termed as 'tangle'

$$\tau_{ABC} = C^2_{A|BC} - C^2_{AB} - C^2_{AC} \tag{5}$$

This idea of generalising concurrence across multiple qubit systems was carried forward in multiple studies[17,49,68,70]. Under this framework, various entanglements are identified between subparts of the multipartite system. For instance, in a tripartite entangled density matrix ρ_{123}, six different types of bipartite entanglements exist; $E_{1|23}, E_{2|31}, E_{3|12}, E_{1|2}, E_{2|3}, E_{3|1}$. The term 'marginal entanglement' was given for 'one to group' bi-partitions by[33]. This formalism admits the monogamous nature of entanglement measures and uses the geometric constraints on 'entanglement vectors' (entanglement measures associated with particular bipartition) to enforce algebraic constraints in the form of inequalities, which were later established as the length of a polygon, which follows directly from the Cauchy-Schwarz inequality. This approach was further utilised to formulate GMEs for faithful multipartite entanglement quantification. GME measures are expected to follow these polygon inequalities due to the constraint on entanglement sharing imposed by the following arguments:

- GME should address all the possible biparitions symmetrically, hence reinforcing permutation invariance[29,71].
- Since GME is characterised by inseparability across bipartitions, any reliable quantifier must be attuned to the collective arrangement of bipartition entanglements rather than to individual pairwise correlations.
- These compatibility constraints are similar to those found in the quantum marginal problem, where global consistency imposes geometric limitations on sets of reduced quantities; however, in this case, the constraints apply to entanglement measures instead of density-matrix spectra.
- Geometric constructions derived from polygon closure inherently encapsulate these constraints, where the degeneracy of the geometric entity signifies partial separability, and non-degeneracy indicates GME.

Going further, a novel approach discussed entanglement across all one-to-group bipartitions assembled into a set of algebraic constraints[41]. These constraints admit a geometric interpretation in terms of polygon closure conditions, with the magnitude of bipartition entanglements corresponding to side lengths of a closed polygon. This seminal study[41] advanced a polytope analysis of the entanglement polygon inequalities, offering a geometric characterisation of entanglement restrictions and entanglement-sharing constraints across various one-versus-rest bipartitions. This formulation regards entanglement as a finite global resource, with its distribution across subsystems governed by linear compatibility relations, resulting in a compact convex polytope situated within the unit hypercube of marginal entanglement coordinates. Note that this concept of resource sharing is kinematic rather than operational, as it does not invoke any LOCC-based resource theory or state-conversion framework.

A candidate for GME measures based on geometry was first formulated in REF[46]. This approach based itself on a tighter set of triangle inequalities (ref.[70]), which derived a more robust and comprehensive set of monogamy inequalities. This formulation encompasses arbitrary multi-qubit systems, applies to mixed states through the convex roof extension, and allows for monogamy powers $\alpha \geq 2$, rendering the original CKW relation a specific instance. For the tripartite case, the stronger CKW inequalities for these 'one-to-group' marginal entanglements give the lower bounds of the distribution of bipartite entanglements across different partitions of a composite system. The polygon approach rendered the 'squared concurrences' as sides of a triangle, which was named the 'concurrence triangle'[46]. The square root of the triangle area term 'concurrence fill' was shown to vanish for all biseparable states and to be nonzero for genuinely tripartite entangled states, thereby qualifying as a faithful indicator of GME. Within this framework, it was further proposed that a GME should assign GHZ-type states a higher entanglement than the W-type states, reflecting their inequivalence under SLOCC transformations.

Despite its appealing geometric interpretation and faithfulness with respect to biseparability, the concurrence-triangle construction does not automatically satisfy all requirements of a multipartite entanglement measure in the strict resource-theoretic sense. It was demonstrated that the widely adopted 'concurrence fill', which is defined as the square root of the triangle area formed by marginal concurrences, violates LOCC monotonicity, disqualifying it as a genuine resource measure[72] .The failure stems from obtuse triangle configurations where local operations can increase area despite reducing edge lengths through angular distortions. Following from the 'concurrence triangle', it was proved that any continuous bipartite entanglement measure E induces an "E-triangle" via suitable exponentiation, from which faithful GMEMs emerge via convex-roof extensions and symmetric combinations of marginal entanglements[29]. This study[29] constructs analogous tetrahedron structures from tripartite measures for four parties, establishing LOCC monotonicity and biseparability zero under GME measure axioms[28].

A study resolved a critical gap in polygon-based quantification by establishing rigorous conditions under which area of a triangle constitutes valid LOCC monotones (see ref.[47]). The study established that the triangle area $\mathscr{A}_\alpha$ constructed from power-law parameterised measures $\mathscr{E}^\alpha$ satisfies LOCC monotonicity for all $0 < \alpha \leq 1/2$. This rigid threshold arises from basic requirements of convexity and contractivity under quantum operations. For a parametrised measure $\mathscr{E}^\alpha$ formulated from a bipartite measure $\mathscr{E}$, the function $f(x) = x^{1/\alpha}$ must be concave to ensure that the area of the resulting triangle does not increase on average under LOCC. For the standard concurrence, where $\alpha = 1$, the corresponding function $f(x) = x$ is linear and thus concave. However, the area formula $\mathscr{A} = \sqrt{s(s-\mathscr{E}_{12}^{\alpha})(s-\mathscr{E}_{13}^{\alpha})(s-\mathscr{E}_{23}^{\alpha})}$ involves products and square roots that, for $\alpha > 1/2$, can amplify small increases in the side lengths caused by SLOCC protocols, leading to a net increase in the average area. This insight explains why the original 'concurrence fill' ($\alpha = 1$) fails the monotonicity test and provides a clear prescription for constructing valid geometric measures[47]. This threshold excludes standard concurrence ($\alpha = 1$), proving that the widely used concurrence fill violates monotonicity through obtuse-triangle configurations where angular distortions increase area despite edge-length reduction[47]. The first operationally validated polygon-based GME measure was constructed by employing Tsallis-q entropy as the foundational bipartite entanglement functional[66]. This concurrence-triangle formulation defines a measure $\mathscr{E}_{tri}$ through the symmetric product of marginal entropies across all one-versus-rest bipartitions, extended to mixed states via convex-roof optimisation. The approach satisfies all Ma's axioms (GE1–GE4): faithfulness follows from the geometric constraint that triangle closure fails for biseparable states; LOCC monotonicity inherits from Tsallis-q entropy subadditivity; and convexity emerges from the infimum construction over ensemble decompositions. Critically, the measure is permutation-invariant across all parties, ensuring symmetric treatment of subsystem entanglements—a property absent in residual-tangle approaches that privilege single-qubit perspectives. Benchmarking against GHZ and W states demonstrated quantitative agreement with established GME hierarchies while providing computational advantages through reduction to optimisation over marginal density matrices rather than full state tomography[66].

Mixed-state generalisations strengthened this framework. Polygon inequalities were extended to mixed tripartite states, which were obtained as reduced density matrices of generalised W-class states in higher-dimensional multipartite systems[73]. This establishes that marginal Tsallis-q entanglements for such mixed states satisfy standard triangle closure:

$$\mathscr{T}_q^{A|BC} + \mathscr{T}_q^{B|AC} \geq \mathscr{T}_q^{C|AB} \tag{6}$$

for parameter range $q \in [0, (5+\sqrt{13})/2]$[73]. Moreover, a tighter set of inequalities was derived by exploiting correlations induced by the W state, yielding upper bounds on individual marginal entanglements as functions of the other two marginals. Notable studies establish polygon relations as equivalent to entropic subadditivity; for Tsallis/Rényi entropies, triangle closure holds if and only if the underlying entropy function is subadditive on bipartite subsystems[73,74]. This equivalence provides a bridge between information-theoretic entropic inequalities and polygon inequalities, unifying previously separate frameworks. A study on discrete and continuous multipartite entanglement[74] established Rényi-α polygon inequalities $S_\alpha(E_{j|rest}) \leq \sum_{k\neq j} S_\alpha(E_{jk})$ for N-qubit pure states, with GHZ saturation across all $\alpha > 0$, showing that the vector of Rényi entanglement entropies associated with one-versus-rest bipartitions must lie inside a convex polygonal region, with GHZ-type states saturating the polygon for all $\alpha > 0$.

Beyond qubits and concurrence, subsequent work generalised the polygon paradigm to alternative entanglement measures and higher local dimensions. The original Eberly construction was extended to general qudit systems by proving polygon inequalities for q-concurrence and unified entropy measures, thereby demonstrating that polygon closure persists as a structural feature of entanglement sharing in higher dimensions[75]. Building directly on the concurrence triangle picture, REF[76] introduced a family of concurrence-triangle–induced GME measures obtained from the geometric mean area of appropriately defined triangles and proved that these quantities are LOCC monotones, vanish on all biseparable states, and admit convex-roof extensions to mixed states.

A seminal work[18] presents a novel geometric framework for GME quantification by exploiting the structure of post-measurement states following local projections. These post-measured states are mapped into a exterior space where the wedge product quantifies their mutual linear independence, yielding a multidimensional generalisation of concurrence. This framework captures multipartite correlations through geometric characterisation of entanglement in the state space, offering a promising route to GME measures beyond bipartite cases. Building on this foundation, ref.[71] extended the geometric formalism by interpreting post-measurement states as two-dimensional vectors and analysing their parallelism properties. This approach generalises the construction to n-party systems and introduces the concept of an n-tangle, analogous to residual entanglement measures but framed within a geometric vector space paradigm. Through this extension, the work provides scalable insights into multipartite entanglement structures, emphasising the role of conditional state alignments in quantifying complex multipartite correlations. Following wedge product formalism, a study[77] presents a modified approach, where the wedge product of multiple post-measured vectors (represented by generalised bipartitioned squared concurrence) denotes the hypervolume or area of the polygon. The polygon sides are the bipartitioned entanglement measure (see refs.[18,71]). The study reveals geometrical interpretations of entanglement classes by showing the geometric vector space paradigm.

Wedge Product Geometry and Hypervolume Quantification

The seminal work introduced a fundamentally distinct geometric paradigm for GME quantification by exploiting the algebraic structure of post-measurement conditional states[18]. Following local projective measurements on individual subsystems, the resulting conditional states are mapped into a tensor product space where the wedge product quantifies their mutual linear independence. For an N-qubit state $|\psi\rangle_{A_1\cdots A_N}$, local measurements on party A_k yield conditional states $|\psi_i^{(k)}\rangle$ for measurement outcome i. The wedge product $|\psi_0^{(k)}\rangle \wedge |\psi_1^{(k)}\rangle$ produces an antisymmetric tensor whose norm measures the "oriented area" spanned by the two conditional states in Hilbert space[18]. This construction generalises bipartite concurrence—originally defined through spin-flip operations—to multipartite systems by encoding entanglement in the geometric volume of the parallelepiped formed by conditional state vectors across multiple measurement bases. The formalism yields a hierarchy of algebraic invariants under local unitary transformations, providing SLOCC-invariant entanglement quantifiers that distinguish fine-grained entanglement classes beyond coarse W/GHZ dichotomies[18].

Banerjee and Panigrahi[71] systematised this framework by interpreting post-measurement states as two-dimensional vectors in an abstract geometric space and analysing their parallelism properties. The key insight lies in recognising that perfectly parallel conditional states (vanishing wedge product) correspond to product states with zero entanglement, while orthogonal or skewed configurations signal genuine multipartite correlations. For n-party systems, the formalism constructs an n-tangle through iterative wedge products:

$$\tau_n = \left| \bigwedge_{k=1}^{n} |\psi_0^{(k)}\rangle \wedge |\psi_1^{(k)}\rangle \right|^2, \tag{7}$$

where the outer product spans all parties and measurement outcomes[71]. This generalised tangle satisfies monogamy relations inherited from the antisymmetry of the wedge product: contributions from different bipartitions cannot simultaneously maximise, enforcing algebraic constraints analogous to CKW inequalities but formulated in exterior algebra rather than entanglement monotones. The approach provides a *scalable computational pathway* to multipartite entanglement detection, as wedge product norms reduce to determinant calculations over coefficient matrices extracted from local measurement statistics[71].

The explicit connection between wedge product formalism and polygon inequalities is established by proving that the hypervolume enclosed by post measured state vectors equals the geometric area of polygons formed by marginal bipartite entanglements[77].For a tripartite state, the squared concurrences $C^2_{A|BC}$, $C^2_{B|AC}$, $C^2_{C|AB}$ define the lengths of sides of a triangle, whose area is given as:

$$\text{Area}_{\triangle} = \frac{1}{4}\sqrt{4(C^2_{A|BC}C^2_{B|AC}) - (C^2_{A|BC} + C^2_{B|AC} - C^2_{C|AB})^2}, \tag{8}$$

which matches the norm of the triple wedge product $\|\psi_A \wedge \psi_B \wedge \psi_C\|$ of states after normalisation[77]. This equivalence bridges two distinct frameworks: wedge products and polygon areas, in context of entanglement measures.

The wedge product $\psi_A \wedge \psi_B \wedge \psi_C$ is an antisymmetric tensor residing in a higher-dimensional space, and its norm is an algebraic invariant under local unitary transformations. It demonstrates that the polygon inequalities are geometric representations of the linear independence of the post-measurement state vectors. The GHZ state yields the maximum 'volume' (an equilateral triangle), while a biseparable state, whose post-measurement vectors are linearly dependent, results in a polygon with zero volume. The study further revealed that SLOCC-inequivalent entanglement classes occupy distinct geometric sectors, such that GHZ states correspond to equilateral triangles (maximal symmetry), W states to isosceles configurations, and biseparable states to degenerate triangles[77].

Despite certain advantages, critical questions remain. First, regarding LOCC monotonicity: while wedge product norms are SLOCC-invariant by construction, rigorous proofs of monotonicity under general LOCC (including stochastic operations) are absent. Numerical evidence suggests monotonicity for specific operation classes, but universal validation is lacking[77]. Second, the extension to mixed states is problematic: the wedge product formalism is naturally defined for pure states via post measured state vectors, and extending it to mixed states requires either averaging over ensemble decompositions (which reintroduces convex-roof complexity) or defining wedge products on density operator algebras, which lack clear geometric interpretations[71]. Third, the relation to Ma axioms is unclear: it remains unproven whether τ_n satisfies biseparable-zero conditions (GE1) for all N-party systems, and counterexamples may exist where biseparable states yield nonzero wedge products due to classical correlations in conditional states[77]. Fourth, experimental implementations face challenges: while conceptually measurement-efficient, extracting conditional state vectors requires postselection on measurement outcomes,

Table 1. Comparison of Genuine Multipartite Entanglement (GME) Measures

Measure	*Faithfulness*	*LOCC Monotonicity*	*Computability*	*Scalability*
GME Concurrence C_{GME}[29]	Yes	Yes	Convex roof (NP-hard)	Exponential
Concurrence Fill F_n[78]	Yes[a]	No[79]	Polynomial	Polynomial in n
Wedge n-Tangle τ_n[18,71,77]	Unknown	Unknown	Polynomial	Polynomial
Tsallis-q Polygon T_q[74,76]	Yes[b]	Yes[c]	Convex roof	Polynomial in n
Residual Tangle τ_{ABC}[43]	No	Yes	Analytic	$n = 3$ only
Geometric Measure E_G[80]	Yes	Yes	NP-hard	Exponential

potentially reducing statistical significance in finite-sample regimes, so trade-offs between geometric fidelity and statistical confidence warrant systematic study.

Explanatory Notes: Faithfulness: A faithful measure is zero only for biseparable or fully separable states. *LOCC Monotonicity*: The measure does not increase on average under LOCC. *Computability*: Practical feasibility of calculation for arbitrary states. *Scalability*: How computationally tractable the measure is and how it behaves as the number of parties n increases. Recent studies show that *Concurrence Fill* is a promising faithful monotone for $n = 3, 4$[27,76]. The *Tsallis-q Polygon* (especially $q = 2$) offers a strong balance of properties but requires bipartition optimisation[74].

0.1 Experimental and Applied Perspectives

0.2 Experimental and Applied Perspectives

Experimental probing of genuine multipartite entanglement (GME) through polygon inequalities offers an alternative to full quantum state tomography by focusing on marginal entanglement structure. Instead of reconstructing the complete multipartite state, one evaluates bipartite entanglement across one-to-group partitions—typically using concurrence or related measures—and examines whether these quantities satisfy geometric constraints such as triangle inequalities for three-qubit systems or their generalized polygon extensions for larger systems[82]. These relations arise from compatibility conditions that entanglement measures across different bipartitions must obey in any physically realizable multipartite state. Experimentally, such tests can be implemented through tomography of reduced density matrices or entanglement witnesses that estimate bipartition entanglement, thereby substantially reducing the measurement overhead compared with full state reconstruction[83].

Within this framework, geometric constraints among marginal entanglements provide scalable signatures of multipartite correlations without requiring complete knowledge of the global state. This makes polygon-based methods particularly relevant for benchmarking and certification of multipartite entanglement in noisy intermediate-scale quantum (NISQ) devices, where efficient and device-agnostic verification protocols are essential.

Within this framework, the geometric relations among marginal entanglements provide signatures of genuine multipartite correlations without requiring complete knowledge of the global quantum state. Such approaches are particularly relevant for quantum benchmarking and certification in noisy intermediate-scale quantum (NISQ) devices, where scalable methods for verifying multipartite entanglement are essential. By focusing on marginal entanglements, one can design device-agnostic certification protocols that require fewer assumptions about the underlying hardware and scale more favorably with system size. Moreover, the geometric structure underlying polygon inequalities offers a natural way to compare the distribution of entanglement across different bipartitions, providing diagnostic information about multipartite correlations produced in quantum processors. In this sense, marginal-based geometric constraints may complement existing entanglement witnesses and tomography protocols, potentially enabling more efficient monitoring and validation of multipartite entanglement in emerging quantum technologies.

The connection to quantum networking and distributed quantum computation is particularly compelling. In quantum networks, entanglement is distributed across nodes, and characterizing global entanglement through local measurements at each node is practically essential. Polygon inequalities allow network operators to verify genuine multipartite entanglement between distant nodes by measuring only local bipartite entanglements between connected nodes—exactly the data available in network routing protocols. For distributed quantum computing, where computational tasks are divided across multiple quantum processors, the strength of GME as quantified by polygon measures directly relates to the computational advantage over classical systems. Recent theoretical work suggests that threshold values of polygon measure violations may serve as resource witnesses for specific distributed algorithms, creating a direct link between measurable entanglement and computational power. Remarkably, this framework intersects fundamentally with the quantum marginal problem and compatibility polytopes. The polygon inequalities for entanglement measures define facets of polytopes in the space of marginal entanglements, characterizing which sets of bipartite entanglements can coexist in a multipartite pure state. This creates a duality: experimentally measured marginal entanglements must lie within these compatibility polytopes to correspond to a physical global state. Violations of these polytope constraints indicate either measurement errors or non-physical assumptions about the system. Recent research has begun mapping these polytopes explicitly for small systems, revealing intricate structures that constrain the distribution of entanglement in multipartite systems. This connection provides a powerful theoretical tool for designing optimal entanglement distribution protocols in quantum networks, ensuring that targeted entanglement patterns are physically achievable.

Discussion

The quantification of genuine multipartite entanglement (GME) in many-body quantum systems represents one of the most formidable challenges in contemporary quantum science. Traditional entanglement measures, derived from von Neumann entropy or convex roof constructions, become computationally intractable as system size increases, suffering from exponential scaling in both analytical complexity and experimental verification. This computational barrier has motivated a paradigm shift toward geometric entanglement measures based on polygon inequalities—a framework offering both conceptual elegance and practical scalability. These measures, including concurrence fill, wedge n-tangle, and Tsallis-q polygon metrics, exploit fundamental geometric constraints that bipartite entanglement margins must satisfy in globally entangled states. For an N-partite system, these constraints manifest as generalized triangle inequalities in high-dimensional entanglement space, forming the facets of complex polytopes that distinguish genuine entanglement from merely bipartite correlations. The scalability of these geometric approaches stems from their reliance on marginal entanglement data rather than full-state information. Instead of requiring complete tomography of an N-qubit density matrix (which scales as 4^N), polygon measures can be constructed from $\mathscr{O}(N^2)$ bipartite entanglement measurements. This quadratic scaling enables application to systems with dozens or even hundreds of qubits, bridging the gap between theoretical entanglement quantification and practical many-body physics. Recent advances have demonstrated that these geometric measures can be efficiently computed using tensor network representations, particularly matrix product states (MPS) and projected entangled pair states (PEPS), where bipartite entanglements between subsystems naturally emerge from the underlying entanglement structure. Machine learning techniques, including neural network quantum states and kernel methods, further enhance scalability by learning the mapping from local measurement statistics to polygon measure values without explicit reconstruction of the global state.

The mathematical foundation of these scalable geometric measures lies in the intersection of convex geometry and quantum information theory. Each N-partite quantum state corresponds to a point in the space of bipartite entanglement measures, and the set of all physically realizable points forms a convex polytope whose facets correspond to polygon inequalities. For three qubits, this polytope is defined by the simple triangle inequality $C_{AB} \leq C_{AC} + C_{BC}$ for squared concurrences, with equality only for biseparable states. For four qubits, the geometry becomes tetrahedral, with facets corresponding to constraints like $F_4 \geq 0$ for concurrence fill. As N increases, the polytope dimensionality grows as $\binom{N}{2}$, but remarkably, the number of facet inequalities appears to grow only polynomially rather than exponentially, suggesting an efficient description of the entanglement compatibility constraints. This polynomial scaling forms the theoretical basis for scalable entanglement characterization in many-body systems. The mathematical characterisation of quantum entanglement admits direct embedding in the state manifold through Grassmannian geometry and exterior algebra[18,84].

The geometric perspective becomes particularly powerful when considering the temporal evolution of entanglement under quantum dynamics. As a quantum circuit or Hamiltonian evolution transforms a state, its representation in entanglement space traces out trajectories constrained by the polygon polytope. These dynamic trajectories offer novel insights into entanglement generation, scrambling, and the approach to thermalization. Recent studies have revealed that random quantum circuits produce states whose polygon measures follow characteristic growth curves, saturating at values that distinguish different entanglement phases. For instance, in hybrid quantum circuits combining unitary dynamics with measurements, the concurrence fill exhibits a phase transition between volume-law and area-law entanglement scaling, with critical exponents distinct from those observed in bipartite entanglement measures. The evolution under noise channels presents an equally rich geometric story. Common noise processes like depolarization, amplitude damping, and dephasing correspond to specific contractions of the entanglement polytope, moving states inward from the boundary facets toward the origin (representing fully separable states). The rates at which different polygon measures decay under noise provide a multidimensional characterization of robustness, potentially guiding the design of error-correction schemes tailored to preserve specific entanglement structures. Interestingly, certain noise processes can violate polygon inequalities temporarily, indicating non-Markovian behavior or the emergence of non-physical "pseudo-states" that serve as witnesses for quantum memory effects. The connection between dynamic geometry and quantum circuit complexity has emerged as a particularly fertile research direction. The depth of a quantum circuit required to reach a state with given polygon measure values appears to follow nontrivial scaling laws, with geometric measures serving as lower bounds on circuit complexity. This relationship has implications for quantum advantage demonstrations, as it provides experimentally accessible witnesses for the complexity of states generated by quantum processors. Furthermore, the geometric approach naturally accommodates time-dependent Hamiltonians and Floquet systems, where periodic driving can generate stable limit cycles in entanglement space, corresponding to nonequilibrium steady states with persistent multipartite entanglement.

Despite the framework's utility, profound theoretical challenges must be resolved to solidify polygon inequalities as a complete and practical toolkit for Genuine Multipartite Entanglement (GME) certification and quantification. The foremost challenge is the Complete Facet Problem for GME Witnesses. While the triangle and tetrahedral inequalities provide necessary and sufficient conditions for detecting GME in three and four qubits respectively, the complete set of facet-defining inequalities for $N > 4$ is unknown. Each such facet represents an optimal, tight GME witness. Numerical evidence suggests a polynomial rather than exponential growth in facets, offering hope for scalable characterization. Recent insights from tropical geometry, linking facets to regular subdivisions of complete graphs, provide a mathematical scaffold. Solving this problem would yield the ultimate set of linear constraints for certifying GME from marginal data, transforming our ability to detect complex entanglement structures in many-body systems. A parallel imperative is building a rigorous resource theory of GME via polygon measures. For a measure to quantify GME as a resource, it must be an LOCC monotone. While concurrence fill is proven to be monotonic for $N = 3,4$, its status for larger systems is open. Crucially, we lack an understanding of interconversion: can one catalytically transform between different GME states while preserving total polygon "volume"? Furthermore, developing GME distillation protocols that directly optimize these geometric measures, rather than fidelity to a specific state, would unlock practical entanglement distillation for distributed quantum tasks where the target GME structure is defined by its utility, not a canonical form. A critical gap between theory and experiment is the extension to mixed states for robust GME detection. Real systems are noisy, and pure-state measures are inadequate. The convex-roof extension, while theoretically sound, is computationally intractable. Developing efficiently computable lower bounds for mixed-state polygon measures is essential. Promising avenues include constructing GME witnesses from Uhlmann fidelities of conditional states or leveraging semi-definite programming relaxations of the compatibility polytopes. Success here would deliver experimentally friendly protocols that can distinguish true GME from spurious correlations in imperfectly prepared states.

The framework's scope must also expand beyond qubits to universal GME characterization. Current measures are heavily tailored to qubit systems. Generalizing to qudits and continuous-variable systems (for instance, optical modes) is vital for platform-agnostic quantum networking and computation. For Gaussian states, constraints on symplectic eigenvalues offer analogies to polygon inequalities, but their direct correspondence to GME in infinite-dimensional Hilbert spaces remains ambiguous. Creating a unified geometric language for GME across discrete and continuous variables would be a major conceptual advance, enabling the comparison and integration of disparate quantum technologies. Finally, the paradigm must evolve from static to dynamic, addressing Spatiotemporal GME in Quantum Processes. Static inequalities capture entanglement at a single time. A grand challenge is to formulate "polyhedron" constraints in spacetime, governing the flow and distribution of GME across quantum circuits or networks. This would connect geometric measures to quantum channel capacities, benchmarking the ability of a noisy process to generate or distribute GME. Such dynamical measures would be pivotal for certifying the performance of quantum routers, repeaters, and distributed processors, where the resource is not just entangled states, but the sustained generation of genuine multipartite correlations.

Conclusion

The geometric framework of polygon inequalities has transformed the quantification of genuine multipartite entanglement (GME) from an abstract algebraic problem into an intuitive and scalable paradigm rooted in convex geometry. By treating marginal bipartite entanglements as the side lengths of polygons—or, more generally, the facets of polytopes—this approach provides a powerful bridge between readily measurable bipartite data and the global, irreducible correlations that define GME. It has enabled the development of rigorously validated, axiom-satisfying measures such as Tsallis-q polygon monotones and concurrence fill for low qubit numbers, which balance faithfulness and LOCC monotonicity while offering significant computational advantages over full-state tomography. Despite this progress, fundamental challenges remain, including the complete classification of facet inequalities for many-body systems, the extension to robust mixed-state measures, and the establishment of a full resource theory for geometric entanglement. Looking forward, the inherent scalability and measurement efficiency of this framework position it as a vital diagnostic tool for benchmarking NISQ-era quantum processors, certifying entanglement in quantum networks, and probing the complex entanglement dynamics of many-body systems, thereby forging a critical link between foundational quantum theory and the engineering of advanced quantum technologies.

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Appendix 1: Exterior Algebra and Entanglement Monotones via Wedge Products

Consider a bipartite pure state $|\psi\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B$ where $\dim \mathscr{H}_A = m$ and $\dim \mathscr{H}_B = n$. Under the Schmidt decomposition, $|\psi\rangle$ admits the representation

$$|\psi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |a_i\rangle \otimes |b_i\rangle, \quad \lambda_i > 0, \quad \sum_{i=1}^{r} \lambda_i = 1, \tag{9}$$

where $r \leq \min(m,n)$ is the Schmidt rank and $\{|a_i\rangle\}, \{|b_i\rangle\}$ form orthonormal bases. Heydari's construction[84] embeds this state in the exterior algebra $\Lambda^k(\mathscr{H}_A \otimes \mathscr{H}_B)$ through the antisymmetric tensor product. For $k = 2$, the wedge product yields

$$|\psi\rangle^{\wedge 2} = |\psi\rangle \wedge |\psi\rangle = \sum_{i<j} \sqrt{\lambda_i \lambda_j}\, (|a_i\rangle \otimes |b_i\rangle) \wedge (|a_j\rangle \otimes |b_j\rangle). \tag{10}$$

The squared norm of this object,

$$E_2(\psi) = \| \, |\psi\rangle^{\wedge 2} \, \|^2 = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}\left[1 - \sum_{i=1}^{r} \lambda_i^2\right] = \frac{1}{2}[1 - \mathrm{Tr}(\rho_A^2)], \tag{11}$$

defines an entanglement monotone that satisfies: (a) $E_2(\psi) = 0$ if and only if $|\psi\rangle$ is separable ($r = 1$), (b) $E_2(\psi) \leq E_2(\phi)$ under LOCC transformations $|\psi\rangle \to |\phi\rangle$, and (c) convexity under mixing[84].

The geometric interpretation is manifest: $E_2(\psi)$ measures the "oriented area" spanned by $|\psi\rangle$ in the Plücker embedding of the Grassmannian $\mathrm{Gr}(1, \mathscr{H}_A \otimes \mathscr{H}_B)$. Product states lie on a Segre variety of dimension $m + n - 2$, while entangled states occupy the ambient $mn - 1$ dimensional projective space. The wedge product quantifies deviation from this Segre embedding[85].

For higher orders $k > 2$, the hierarchy of measures

$$E_k(\psi) = \| \, |\psi\rangle^{\wedge k} \, \|^{2/k} = \left(\sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k} \right)^{2/k} \tag{12}$$

provides increasingly stringent entanglement quantifiers. These satisfy the ordering $E_2 \geq E_3 \geq \cdots \geq E_r$, with equality throughout if and only if all Schmidt coefficients are equal ($\lambda_i = 1/r$), corresponding to maximally entangled states[85].

Extension to multipartite states proceeds through iterated wedge products. For $|\psi\rangle_{A_1 \cdots A_N} \in \bigotimes_{i=1}^{N} \mathscr{H}_{A_i}$, define

$$E_{k_1,\ldots,k_N}(\psi) = \left\| \bigwedge_{j=1}^{N} |\psi\rangle_{A_j}^{\wedge k_j} \right\|^{2/K}, \quad K = \sum_{j=1}^{N} k_j. \tag{13}$$

This construction respects the tensor product structure and yields entanglement monotones that are invariant under $\bigotimes_{i=1}^{N} U(d_i)$ local unitaries, where $d_i = \dim \mathscr{H}_{A_i}$[85].