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Math Curricular Framework for the Education of the Gifted and Talented

National Institute of Advanced Studies
Bengaluru, India

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Math Curricular Framework for the Education of the Gifted and Talented

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Foreword

I am pleased to present the Math Curricular Framework for the Education of the Gifted and Talented (EGT). This is the first of several pieces of curricula that will be created to support the children as part of the NIAS National Programme for the Education of the Gifted and Talented.

A scale up of the EGT programme is contingent upon the availability of transferable resources and processes. Some of the most important features in this regard are the selection criteria for the EGT, the curricular framework and pedagogy for mentoring the children of NIAS-EGT and finally the supplementary resource materials that can provide continuous challenge for these high-ability children. The identification protocols have gone through several versions and are continuously subjected to improvement. The research effort is to create a set of protocols that will address the diverse nature of children who are gifted in India. The NIAS-ALC which is into its second year has helped NIAS create a model that can be scaled up.

A creation of content followed by curriculum development for the NIAS-EGT has, however, been developing organically, based upon the needs and interests of the students, availability of resource persons and their areas of expertise. This document is the first among the many that is planned as an on-going effort in the creation of the curriculum for the gifted and talented. The potential for the outreach of such a curriculum is large given the availability of technology support for online learning.

To the best of our knowledge, this is a unique curriculum development initiative, and we have therefore had to conceptualize this exercise from scratch. The late Prof Baldev Raj, former Director, NIAS, who championed the idea of scale-up of the EGT programme, was also the prime mover of the curriculum development initiative. Prof. V.S. Ramamurthy in his capacity as Advisor to the programme, further espoused and helped progress the idea of development of curricular materials that are mapped to the needs and abilities of the EGT cohort. The Advisory Board members, through their guidance provided during review meetings, have helped shape this initiative to create a core set of curricular guidelines, starting with Mathematics. Discussions of the scope of such a project were first carried out in the Advisory Group, wherein the initial concept of the curricular guideline for Math content for the age-group 12-16 was vetted.

Prof. Anitha Kurup, Head of the Education Programme who has spearheaded the National Programme for the Education of the Gifted and Talented in India, envisioned the curriculum development as a structured document which would collect together the most appropriate and reliable openly-available resources, and make them available to the students. Further, the document would illustrate how a topic could be explored, with sufficient ready-to-use materials. The document was envisioned not a text-book but as a resource that could help students and facilitators navigate the available resources, by identifying and providing adequate guidance towards curated content that would meaningfully supplement the available school courses.

Dr.Shailaja D Sharma, a mathematician and math educator, conducted a detailed study of extant syllabi, in order to identify the curricular pathways that would be meaningful for mixed-age-group classes.

Prof. H.S. Mani (previously with Chennai Mathematical Institute) provided a detailed review and comprehensive comments. In addition, he provided excellent additional resources and extended to us the benefit of his personal network of professional mathematicians. As a result, the report has been viewed by at least 3 mathematicians from his network, from India and overseas. We have benefited from all the comments so received via Prof Mani.

We thank Dr. Nithin Nagaraj and Mr. H. Sudarshan who provided valuable inputs. The Math Resource Material is the first such document to be produced under the aegis of the NIAS EGT. We hope that we shall now be able to produce at an accelerated pace the set of new content materials that are envisaged in various other domains of interest to the EGT programme. No doubt there will be new learning, as we roll out the curriculum via our ALC network. All content documents must necessarily be revised from time to time, in order to stay current with the latest practices in pedagogy as well as content. We shall leverage technology to keep the documents current and our first step in this direction is to place our curricula online.

We hope that this document, and the ones that follow, will be welcomed by all the stakeholders of the NIAS EGT programme.

Dr. Shailesh Nayak
Director, NIAS

Acknowledgements

Writing this report has been a sizeable endeavour, mainly because of the need to bridge between actual classroom practices familiar to teachers and students in most schools in India, and idealized practice. Throughout the exercise, we have been inspired and energized by the vision and commitment of the sponsors of this report, and the community of its target users. The idea of preparing a curricular framework to support the teaching of gifted students was inspired by the vision of the late Prof Baldev Raj, then Director of NIAS. The endeavour to put actual resources in the hands of gifted students and their educators has been propelled forward by Prof. V.S. Ramamurthy and the Programme Advisory Committee for the Education of the Gifted and Talented (NIAS -EGT), for whose commitment and support we are indeed grateful.

This report has gone through several levels of scrutiny and feedback. We are grateful to the reviewers for their painstaking efforts and valuable inputs. Dr. Nithin Nagaraj, NIAS and Mr R. Sudarshan gave detailed inputs in the internal review exercise. Prof H.S. Mani, CMI, Chennai, provided thorough feedback in the external review phase. In addition, Prof. Mani funnelled inputs from several other experts.

We are indebted to the vast number of open source content providers we have cited in this report. The high quality material shared freely by them offers a low-cost pathway for students and teachers with high motivation albeit low means.

We also place on record our thanks to the numerous well-wishers and math enthusiasts, including students, who have pointed us to good content, links and examples. Our students are intrepid explorers of the world of knowledge and we derive energy and inspiration from them.

We have made every effort to improve the report on the basis of the reviews and feedback. Any errors or omissions that remain in this report are attributable only to the authors. This is, ultimately, a live document, and we hope to see it updated and further improved as we go forward.

Anitha Kurup
Shailaja D Sharma
H S Mani

1

Statement of Purpose and Scope

The present report is the culmination of an effort to put together a practical learning guide for gifted students of Mathematics and their teachers, so as to anchor the 'idea' of a gifted learning programme in a real, practical set of activities and exploration. The report is also just the beginning of a journey to create practically useful and comprehensive learning guides for gifted Math education for the age group 14-17 years. It is a beginning, because there is a wealth of resources now available to the inquisitive mind, and a sub-selection is essentially subjective and therefore more or less arbitrary. As we go along, we hope to tailor this learning guide further, such that it balances the best from different pedagogies, curricula and strategies.

Of course, any such learning guide must necessarily be a 'live document', in that it needs to be periodically updated to be in step with the times and changing education practices and preferences.

In this first version, we have put together an overview of the different Math curricula available for the target age-group, a summary of the topics that are typically taught at these levels and then provided a range of additional directions, illustrated via a few topics.
The inspiration for this document comes from two sources. The first, of course, is the long-standing Gifted Learning Programme of NIAS, with its years of endeavours and experience in identifying students with exceptional learning abilities, starting from an early age. Having identified such students and set up centres dedicated to nurturing this special talent, the question naturally arose: what is the recommended course of action with regard to these students? What should they be exposed to, for example in Mathematics? What should they devote their limited time to? What are the problems that are most suited to nurturing their natural curiosity, their eagerness to explore and their willingness to put in the effort? This document tries to provide some initial answers to these questions.

The second source is the direct experience of one of the authors in working with especially motivated and mathematically inclined students in various institutions in Bangalore during 2015-17. The students referred to were from the 11-12th grades. Although the teaching work was independently carried out, and not linked to the NIAS Gifted Learning Programme, the experience of working with students from diverse backgrounds, with high mathematical aptitude in common, has directly influenced the selection of recommended resources in this booklet. Some of these experiences are extrapolated in the recommendations made for younger students. We hope to gain more direct experience of younger students via the Advanced Learning Centres (ALC's) which have been already established and the feedback from those experiences will undoubtedly influence the contents of future editions of this course-guide.

How does an educator recognize a 'gifted' student of mathematics? Not just by the marks scored in the final examination! The gifted student is recognized not as much by the answers she/he gives, as much as by the questions she/ he asks. The ability, the self-assurance, to ask a 'good question' comes -- not from approaching learning as a recipient of given, certain, knowledge, but as a seeker of knowledge and problem-solving methods. Asking 'good questions' is the hallmark of a student who really 'wants to know'. 'Good questions' refers in this case to clarifying questions, questions which help clarify the subject matter, also sometimes (though not necessarily) to extend it, or to connect it with other topics or concepts. Gifted students exhibit the ability to stay with a concept and try to clarify it completely for themselves. They are willing to explore on their own, apply their own understanding to solve new or unfamiliar problems. They are eager to learn and usually take others along with them. In the present discussion, we shall have ample opportunity to reflect upon these traits and to constructively respond to them.

The selection of students with a 'gift' for mathematics is not entirely subjective, however. Students are screened into the NIAS ALC's, for example, on the basis of a comprehensive evaluation framework which is based on test performance, self-analysis by candidates and teacher recommendations. Psychometric testing and a combination of self-assessment and assessment by teachers has been successfully deployed to identify students with the motivation, discipline, commitment and enthusiasm for a deeper experience of mathematics. It is found that the role of parents and mentors is critical for the sustained development of these traits, their effective deployment and realization of the ensuing results. It is the combination of these traits and circumstances that is implied by the short-hand expression 'gift'[1].

There are also very many educators who by experience or aptitude become able facilitators of mathematics education. Often, they distinguish themselves by their intensive familiarity with the subject matter and their remarkable ability to solve a problem at sight. Their facility with mathematics inspires their students, who wish to be as comfortable as them in dealing with challenging mathematical problems. These high-ability educators have much to offer in terms of insights into the learning process, for example, on the use of short-hands, mnemonics, algorithmic solutions, etc. These may be relevant for developing curricular material for the gifted learners. At the same time, they may also find it refreshing to see a focus on concepts, expanded scope, problems from applied domains, and additional learning resources from all over the world.

This course-guide is intended for use by the educators and learning facilitators who are associated with high-school level Gifted Learning Programmes in Mathematics. It provides references and model lessons which the user may find helpful. However, it is not a textbook, nor even a comprehensive teaching aid. The examples and resources cited, when used with discretion by the knowledgeable educator/ teacher, may aid in the development of a meaningful and rewarding learning experience for young students with a particular aptitude for mathematics.

[1] A comprehensive evaluation of the traits exhibited by students enrolled into the NIAS gifted learning programme is not herewith attempted, however it is a topic of interest in itself.

2

Framing the Goals for GLP-MCF

Statement of the learning goals

Justification of the goals

- **Based on extant curricula – national and international**
- **Based on college-preparedness**
- **Based on job-preparedness**

What should be the goals of a mathematics curriculum within the ambit of a Gifted Learning Programme targeting learners from the age-group 14-19? The importance of a written curriculum for a gifted learning programme cannot, according to scholars (Joan Franklin Smutny, 2003), be overstated. However, the bulk of literature on curriculum planning for gifted learning targets middle-school or earlier stages. In moving forward to outline a curriculum for more mature learners, our main recourse is to practical experience in working with such learners, and societal and personal goals that have been articulated in this context. We note here that the wide variety of topics of interest to the target group makes it challenging to devise a curriculum that suits all as well as each learner. Noting the strategies adopted by other programmes that target high-ability students, such as the various Math Olympiads, the high level categories under which the problems may be grouped are Geometry, Algebra, Analysis and Combinatorics. Herein, we have used a slightly more generic learning approach and classified the topics under the 6 categories of Counting, Proof, Space, Change and Chance.

All 'gifted programmes' aim to develop the creativity and talent of the targeted learners, give them opportunities to explore, to deepen their own learning and insight into the subject, and to apply their skills of problem-solving to real-life situations, with its inherent ambiguity and inexplicitness. In addition, in the national and social context in India, a gifted learning programme must also address national and personal aspirations, opportunities and strategies for growth and impact. Some of the specific parameters in this context would include inclusion, overcoming of linguistic barriers, or indeed, leveraging of linguistic and other types of diversity, application to local problems and priorities, benchmarking with international standards, and so on. An additional concept is the incorporation of a wide set of stakeholders, in particular parents, into the learning and growth strategies.

Explicitly stating and acknowledging programme goals helps clarify the strategies and pathways adopted. Such clarity is required not only by the educators involved, but also by the learners themselves, as well as other stakeholders including parents, programme sponsors, and so on.

The present attempt at developing a framework document for math curriculum is predicated on the following goals:

To assist gifted learners of mathematics in their exploration and mastery of the subject in such a way as to:

- Inspire them to develop a deep affinity for the subject and to frame their personal quest in the domain
- Clarify and strengthen foundational concepts which may have been overlooked at an earlier stage
- Develop an understanding of how science develops, how discovery is made, by encouraging them to pursue discovery and providing scientific training
- Expand their mathematical horizons by going beyond the school curriculum
- Develop and practice research and self-study skills
- Expose them to multiple approaches to problem-solving
- Sensitize them to the national heritage in mathematics with the aim of developing awareness and self-confidence
- Practice with challenge problems and puzzles
- Practice with contemporary technology and use of computational tools
- Exposure to current mathematical topics which the school syllabus may not have caught up with (e.g. Statistics, Computer Science)

The large national secondary and high school certification boards in India are the CISCE and CBSE. The goals of the CISCE Math Curriculum (CISCE) at the secondary level (Std X or ICSE certification) are necessarily broad and aim to create basic interest and skills in Mathematics. The CBSE Board (CBSE) establishes curricular goals at the curriculum level and at the subject level. For the Math secondary level curriculum, the goals are similar to those for ICSE, but are more specifically articulated, for example, 'understanding the principles of reasoning and problem-solving'. The goals are articulated at a higher and more comprehensive level.

In terms of the curriculum itself, due to the vastness of syllabus, considerable amount of material is compressed into the school term. This reduces the opportunity for the student to reflect intensively, or to discover results for themselves.

The Cambridge (Cambridge Assessment) system focuses on Math as a 'key life skill' and in order to set a foundation for further mathematical studies. The latter is an unstated goal also of the ICSE and CBSE syllabi. The syllabus objective is also, however, to 'build learners' confidence' by imparting competence and 'fluency' with mathematical concepts. The idea of fluency suggests seamlessness, or easy application to problems. Here the focus is on 'fluency', rather, perhaps, than on intricacy. Interpretation of results and application to real life problems are other highlights.

The International Baccalaureate Middle Years Programme (MYP) (IBO) addresses the learning needs of students from age 9 to 16. The Math curriculum for this programme has three different objectives: to establish a foundation; to prepare students for higher math studies; and to prepare students to use math in the diverse (non-mathematical) areas that they may work in, as well as in everyday life. Students may take the basic math course or a more advanced syllabus. In addition to thinking and research abilities, the curriculum promotes communication skills and social contextualization.

The curricular scope of the certifications varies, and for illustrative purposes and to establish the hierarchy of topics, a spread-sheet has been provided further on. However, individual boards may have significant divergences from this hierarchical representation. For instance, the IB curriculum introduces Statistics and Probability as a module in the 'middle years programme' (MYP), alongside of Number, Algebra and Geometry/ Trigonometry and the Cambridge IGCSE syllabus introduces Vectors and Linear transformations this (middle) stage. There seems to be a clear shifting towards discrete mathematics, combinatorics and statistics in the overall curricular trend.

Taking from all of the above, the goals of a Gifted Math Curriculum for the Secondary Level (age group 14-16) should ideally include, in an age-appropriate manner:

- Develop an appreciation for the inter-dependencies in various branches of science
- Be able to recognize patterns
- Frame problems from a situation/ mathematical modelling
- Communicate mathematical results

With regard to high-school mathematics, once again, we may refer to the 4 curricular programmes mentioned earlier. In this case, the references shall be to the ISC certification, the CBSE certification, Cambridge AS and A-Level certification and the IB Diploma, including the HL (High Level) Math syllabus.

The ISC Math syllabus seeks to enhance analytical and rational thinking skills, computation, generalization, logic, and use of charts, graphs, etc. The CBSE syllabus focuses on creating the ability to use tools and methods in different situations correctly. In terms of syllabus coverage both are similarly vast.

The Cambridge exams focus on equipping the student with skills for effectively using mathematical information, modelling, considering accuracy requirements in context, think logically, analyse results and reflect on findings. The syllabus is modular, allowing students to choose mechanics or drop it.

The International Baccalaureate (IB) Diploma optional module titled 'higher level' (HL) math focuses on the twin goals of mathematical rigour and application. Independence in mathematical learning is emphasized. Apart from the goal of transferability of the knowledge to other disciplines, the curriculum uniquely emphasizes 'clear and confident' math communication as well as of the moral, social and ethical dilemmas that arise in the application of science (including math). This latter goal is unique to this programme.

Taking from the above, the goals of the Gifted Learning Programme for the age group 14-19 would benefit from incorporating the following goals:

- Ability to work independently in Mathematics
- Skill in Math communication
- Ability to reflect on findings and interpret them
- Awareness of the ethical dimensions of the applications of Math & Science

A curriculum exercise must also, in the context of national and social priorities, address itself to college-readiness and job-readiness.

Currently, college-readiness in the Indian context often means high versatility in problem-solving of the variety that is seen in competitive examinations. This remains a high-stakes focus for high-ability students and should not be neglected. Thus, exposure to large numbers of challenging 'typical' problems from the competitive examination catalogues should indeed find a place in the syllabus. "How IIT entrance exam can be beaten" remains a typical theme for high-ability aspirants.

However, restricting to this type of competitive problem-solving would be inappropriate. Increasing numbers of colleges in India, and most of the colleges in preferred higher education destinations in Western countries seek analytical, interpretational and communication abilities.

Students also find themselves better able to determine their course of future studies if they are well-exposed to different domains in which Mathematics is applied. For example, students may choose engineering professions, if they enjoy working with differential equations, or computer science, or indeed biology or finance, if they are fascinated by linear algebra, or data science or the humanities, if statistics really appeals to them, and conversely. Students who understand the applicability of mathematics to different domains have a greater advantage when it comes to selecting their areas of future specialization, and vice versa. For example, a student with a pre-disposition towards aeronautics may need to be particularly exposed to the use of calculus in framing and solving the equations of fluid dynamics.

A gifted learning programme may also offer the space and time for the student to explore and practice the use of tools and technologies. In one example, students worked on fractal shapes, using a software package that was available on the internet, in the process learning and reinforcing concepts in recursive functions, geometry of scale and probability. Fractals are not a topic in school math curricula, but are a great topic for exploration.

Students may also set themselves projects or experiments, for example statistical experiments, to collect and analyse data, using tools that they have only read about in their textbooks.

The job market also values specific skills in the young graduate, for which the young gifted learner may be prepared. These include for example the ability to visualize a problem, or to translate a situation into a problem statement and then bring to bear upon it the tools of mathematics. In one anecdotal example, a large oil and gas company had to train each one of its young engineering graduates how to make a sketch of a problem situation, before solving it. This was a problem in math communication, as much as in analysis and problem-solving. The habit of clearly communicating math is usually highly under-rated and not emphasized in typical school and college education. The gifted learning programme can aim to bridge that gap.

Yet another possibility is to prepare and collect problem scenarios from different domains, be in aeronautics or genetic engineering, with a number of sub-problems for the students to ponder on. Development of such problems and solutions requires focused input by specialists and is thus input-intensive, but would go a long way in addressing job-readiness and career-readiness requirements in the curriculum.

Thus, college-selection and career-selection presents a set of goals as follows:

- Familiarity with various applications of mathematics and opportunities to explore
- Ability to do independent work, including research, in a chosen area
- Familiarity with and practice using various technologies and computational tools
- Ability to structure, visualize and analyze a real-life situation via sketches and communicate it to a third party

The above discussion provides a fairly comprehensive set of goals for a gifted learning programme. In reality, each intervention must typically connect with one or other of these goals and the overall programme must be so structure as to propel the learners in the direction of each of these goals.

3

Pedagogical Strategies & Resources

Overview of future-readiness

Critiques of curricula and pedagogies

- **Comparison of extant syllabi**
- **Critique and Recommendations**

Extant curricula and related pedagogical approaches have been touched upon in the previous section, therefore we shall now discuss in greater detail the pedagogies relevant to gifted learning.

A curriculum for gifted learning has to be anchored at three points (Joan Franklin Smutny, 2003): advanced, in-depth knowledge of course content; higher order thinking skills and discipline-specific processes of locating, applying, evaluating and creating new information; and a focus on real-life applications and issues related to the topic of study. Students, learning the content, skills and processes associated with the discipline, as a consequence, should become creative producers and contributors to it.

The curriculum has to be centred on course content which is mapped to the syllabus that is taught at their grade level in school. Evaluation of the student's progress has to be linked to performance with respect to this core syllabus. The gifted learning programme has to achieve a combination of core syllabus and enrichment material. Taking the student away from the core syllabus, may be detrimental to the student's progress, given the school-centric context of the gifted learning model that the present curriculum is premised upon. The purpose of the gifted learning programme is to achieve mastery over the school curriculum, while expanding scope to accommodate many topics that may not be included in the school curriculum, as well as providing avenues for independent exploration, discovery and fulfilment.

Thus, explorational topics shall be identified based on the students' current learning and topics that are taught in the core school syllabus. The point of take-off should be the achievement of school syllabus learning goals.

As an example, if a student of the 10-12th grade is studying Vectors, they may have fundamental questions about whether vectors can be moved about in Cartesian space. They may have questions about the difference of two vectors, and the point upon which that new vector is acting. The concept of position vectors and the equivalence of each vector to a unique position vector is usually not fully explored in the classroom. Not every classroom has a teacher with sufficient expertise, nor does the syllabus provide the time for students to necessarily undertake such reflection. The gifted learning centre provides the space in which many such questions can be explored, the implications reflected upon and results more completely assimilated.

This discussion requires some sketching and visual aids to assist in understanding and this could be the topic of a whole session for a gifted math learning class.

A typical question in this regard is: Does the following represent a line, a plane or a point?

$$ax + by + cz = d$$

The question comes about because the equation is reminiscent of the (2-dimensional) equation of the (one-dimensional) line and a simplistic intuition may be that the equation defines a unique line in 3-d space. Of course, the answer is that it defines a (2-dimensional) plane in 3-dimensional Cartesian space, or in other words, all the points (x,y,z) that fulfil this equation, taken together define a plane. However, if two such equations in 3 variables are provided, they refer to a (one-dimensional) line in 3-dimensional Cartesian space, being the intersection of two planes, or the set of points (x,y,z) that fulfil both the equations. And if there are 3 such equations, they may refer to a line, or a point, if indeed the corresponding planes do so intersect, but since they may not all intersect in a common line or a common point, the 3 equations taken together may not give any tangible solution (and would then be said to be inconsistent).

Students may ask questions like 'what does it take to describe a non-planar surface in 3-dimensional space' and they may look for analogies in the 2-dimensional space. These are good conversations to have in this class. With the aid of the internet and a willingness to explore together, teachers and students could profitably expand their own understanding and insight into algebra and non-Euclidean geometry through this dialogue.

One of the major 'gaps' from the regular classroom, that the gifted learning class fills, is the gap of 'reflection'. Gifted learning classrooms have to be able to provide the all-important 'time for reflection' to the learner. This vital ingredient is significantly missing in the mainstream Indian curricula, forcing students into the mode of rote or mindless learning. Here, students must learn mindfully, with awareness about the findings and methods, as well as about the learning process itself.

Problem-based learning is also well-suited to the needs of gifted learners at all levels, although it has been discussed in the literature mainly in the context of elementary level learning. In this scenario, the learner is presented with a loosely or incompletely structured problem, replete with uncertainties or vagueness, as is often the case in real-life situations. The students work with such information to frame the problem, establish simplifying assumptions, research the problem if required, draw from analogous problem situations, and so on, and develop the approach to 'handle' if not completely 'solve' the problem.

An example from a Std 11-12 curriculum would be, for example to determine how many distinct solutions there might be to the problem of partitioning a given whole number. Students would be welcome to gain initial familiarity and insight into the problem by working inductively with small numbers. They may discuss and research methods developed by others, for example they may want to refer to Ramanujan's work on partitions of whole numbers. They may want to discuss whether or not negative numbers and zero's should be allowed. They may use analogies from statistics, or computer science. They may develop combinatorial methods to count the possible ways to arrange n things in r groups. They may need to extend the method to cover the possibility of having zeroes in the partitions. They may start looking at vectors of natural numbers with a constant total and try to assign an interpretation to such vectors, thus entering the domain of block designs.

The steps of such a learning process would be:

1. Frame the problem:

How many vectors $\bar{x} = (x_1, x_2 \dots x_n)$ of length k are possible, such that $\sum_1^k x_i = n,\ x_i \in \mathrm{N}$

2. Develop analogues for the problem:
   This is the same as partitioning a natural number into non-zero, non-negative additive components

3. Develop an approach to the problem:
   We can handle this by treating the number n as a collection of n indistinguishable objects which have to be grouped into r sub-sets

4. Study simpler problems and use inductive thinking:
   Work out the partitions of the numbers 1, 2, 3, 5

5. Arrive at a generalizable result
   $$\begin{matrix} Number\ of\ partitions\ of \\ n\ into\ k\ non-zero\ wholes \end{matrix} = \binom{n-1}{k-1}$$

6. Optionally, extend the formula to include partitions that allow zeroes by using the transformation:
   $$y_i = x_i + 1$$
   to get the result for the less restrictive problem of partitioning n into r non-negative whole numbers:
   $$\begin{matrix} Number\ of\ partitions\ of \\ n\ into\ k\ wholes\ (including\ 0) \end{matrix} = \binom{n+k-1}{k-1}$$

The last step in the above would almost certainly have to be triggered and facilitated by the instructor. But steps 1-5 can be achieved by the gifted learner largely by himself/ herself.

The above illustration shows that the problems for the gifted learning curriculum can be picked from almost anywhere, but must be picked carefully and preferably worked out initially by the instructor. It is not essential that the instructor have prior familiarity with the problem. However, the instructor must be open to an open-ended discussion and sharing the learning journey with the learners.

Gifted students must be encouraged to solve problems independently and taking recourse to any methods they already know or have heard about. They should be encouraged rather than discouraged to learn and use the research techniques of professionals in different fields of study (Joan Franklin Smutny, 2003). Instructors must have enough activities, problems and materials ready for at least 50% of the course schedule. Students may take more or less time to work their way through these prepared materials. The instructor would be better advised to assume that they will progress rapidly through the planned problems and materials, and prepare accordingly.

Active participation and engagement of the students is a table-stake for the gifted learning programme. Here, rather than the instructor presenting new material, the students are expected to be motivated by their own explorations of a given topic. The instructor plays a more subtle role, of guiding and encouraging the exploration, introducing alternative ways to look at a problem, clarifying difficult material and so on.

Meeting guest speakers is also a recommended part of the curriculum. Students must learn from practitioners how they actually use mathematics and statistics to solve problems. Speakers may be from any walk of life, as long as they are able to provide new insights and inspiration to the gifted learners. Contact with these experts[2] brings added depth and dimension to the students'

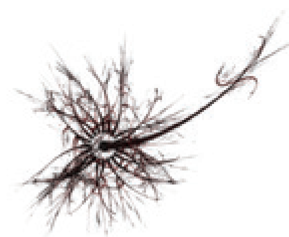
[2] In the experience of the NIAS Advanced Learning Centres (ALC), parents of the students enrolled in the gifted learning programme also serve as an excellent and engaged resource pool.

experiences, especially if they are willing to share their enthusiasm, knowledge and expertise (Joan Franklin Smutny, 2003).

Speakers about mathematics do not have to be math researchers alone. It is said that Math is best taught by those who use mathematics. They could be economists, advertising professionals, football coaches or social media marketing professionals, and so on. Experts can be found on the internet as well as locally. International experts may be willing to join the gifted learning class by video-conferencing. Websites like Khan Academy, Better Explained and Brilliant have charismatic founders and team members, who may be willing to interact with the students. Groups like the Indian Math Olympiad (INMO) organizers, TIFR, NCBS, Azim Premji University's (APU) education department, etc. have expert speakers and practitioners who could be a national resource for gifted learners. Most professional social networks have large numbers of experts – whose expertise can be tapped. Learning advisors or educators have to take the lead in roping the experts in.

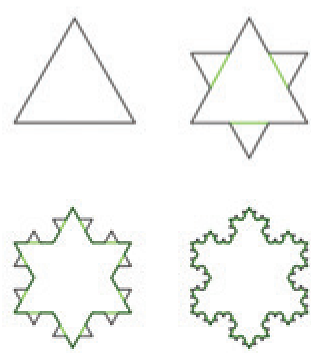
A gifted learning programme must also incorporate project work. One example of possible project work with the age group 14-19 is to make physical models of fractals, for example, the Menger Sponge. In one example[3], a gifted student, with no external prompts, identified a video on YouTube, which demonstrated the method to make such a model using old business cards (see image below, the picture was taken after the model had been considerably man-handled! The original model had no loose components. On top right of the picture is visible an origami 'flower vase' by another student.). The student went on to make the model himself, bringing together his aptitude not only for mathematics, but also for engineering and art.

Another such project example consists of working with so-called Context-free Software (Context Free Art) to create fractal images. The code is easy to learn, and the images are of a high quality and sophistication. The act of creating those images generates questions as well as answers and explorations in the minds of the user. (The image on right is credited in the footnote[4].)



Such a project could be extended by adding readings from Benoit Mandelbrot, from his original work: The Fractal Geometry of Nature (Mandelbrot, 1977). For example, students may be able to tackle the following problem discussed in detail by Mandelbrot:

Coastal Dimension of a Triadic Koch Lake:



A triadic Koch lake, or a Koch snowflake, is a fractal shape created from an equilateral triangle. Each side of the triangle is further trifurcated and the middle section is replaced by an equilateral triangle *whose* side is equal to the trisection of the side of the original equilateral triangle. This process

[3] Credit to Zaki Sista (17 yrs), batch of 2018. Origami flower-vase by Mrinal Ghosh(18 yrs), both from MAIS Bangalore.

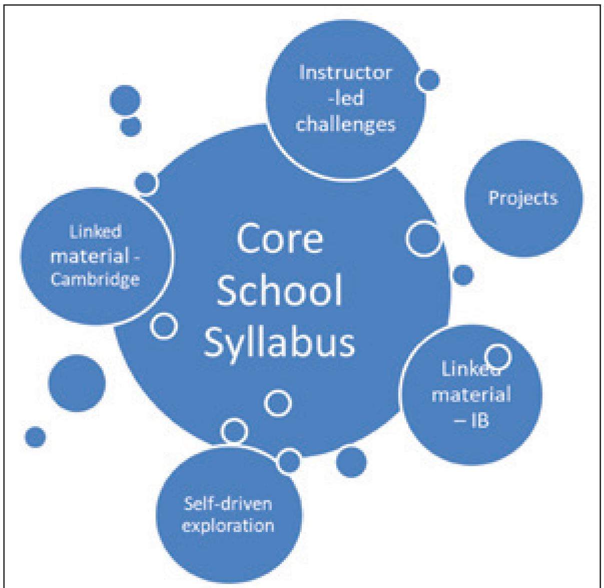
[4] Spliced by efi, Variation: BMY, uploaded on July 12th, 2007.

of trisection and replacement is continued indefinitely, yielding a Triadic Koch Lake.

Students may construct this shape and attempt to determine the length of its circumference, which is obtained as an infinite divergent series.

They may then proceed to determine the shapes arising analogously by quadric Koch curves.

Such a project would not map onto any item in the current curriculum of the students, but they are eminently within the reach of the students and filled with the gems of insight and intelligence that gifted students seek! In addition, it has the visual stimulus that would be an aid to any mathematical project.



Curricular Model:
Gifted Learning - Mathematics

Gifted students in this age-group do not need to be given homework and assignments, as their learning is self-motivated. They will probably undertake a lot of work while not in class, on their own. Evaluation, however, is an important part of the pedagogy, although it is not a straightforward matter of testing on a pre-determined set of learning outcomes. We shall deal with assessment separately.

Several examples of inductive thinking and practice modules in mathematics for younger learners are provided by Christopher M Freeman in Chapter 9 of the book Designing and Developing Programs for Gifted Children (Joan Franklin Smutny, 2003). These are recommended to the reader.

A summary of the pedagogical approach and curricular model for the Mathematics Gifted Learning Programme is provided in the accompanying figure.

Within the framework of this model, the pedagogical objectives and strategies of the Gifted Learning Programme may be articulated as follows:-

1. To fill the gaps, if any, in the students' learning understanding of concepts taught in the regular class

2. To allow for practice, by accessing 'question banks[5]' that offer plenty of material for repetition, thereby generating familiarity

3. To extend the learning, by accessing value-adding contiguous materials from other curricula for the same or adjacent grade levels

4. To expand learning, by promoting exploration to new domains of mathematical curiosity,

5 Question banks with hundreds of solved or unsolved questions are a vital resource for students in the State Boards and National Boards, whereas students of IB and CIE are known to often take recourse to collections of past question papers for practice and exam preparation.

self-propelled by the student

5. To challenge and stimulate, by making available to students challenging problems and interesting questions to explore

6. To foster skill in research methods and independent investigation and problem-solving

7. To expose the student to real-life problems and situations by interactions with professionals and problems from various domains

The pedagogy will emphasize:

1. Practice
2. Reflection
3. Discovery
4. Self-paced exploration and learning
5. Project work
6. Team-work
7. Use of tools and technologies
8. Use of open-ended methods such as discussion, brainstorming, research, trial-and-error, inductive reasoning, intuition, analogy, general reading, etc.

4

Content Organization and Flow

Syllabus mapping across Std IX to XII

Content hierarchy and interdependences

Identification of Additional content sources

Content Flow – Standard Curricula

The mainstream high school mathematics curriculum in India is deemed by public opinion to be of a high standard. The syllabus is rigorous and demanding. It covers a wide scope, but also explores individual topics in depth. A very large number of problems of many types are solved as a part of the study course. Students studying mathematics at the high school level (11^{th} and 12^{th} standard) are expected to work at Mathematics every day and solve 10-20 problems on a daily basis. Students successfully passing through this programme should be expected to be equipped to solve at sight a wide range of problems in mathematics.

Upon reviewing the syllabus content across different syllabi, it was found the broad structure of content was largely similar, although modularization at the higher levels brings about substantial changes in the final high school years. Another point of differentiation is the extent to which visualization is emphasized. In general, Mathematics curricula prescribed by Western examination boards do not seek to cover as much content as the Indian boards do. However, they make up for it in terms of the focus on reflection, insight, analysis and application.

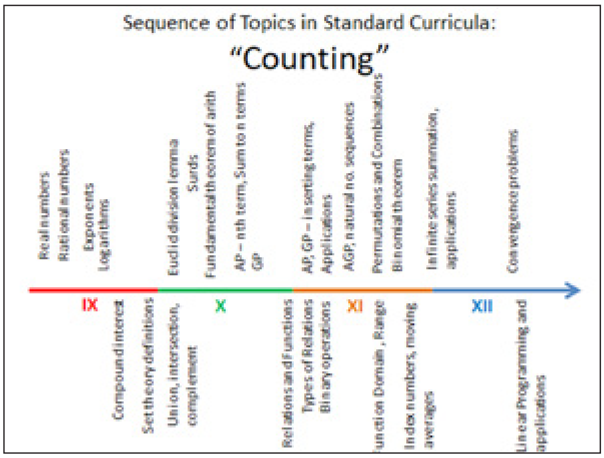
Math curricular content for Std IX through XII in the mainstream high school programmes follows a standard pattern. The following are the main topics that are covered:

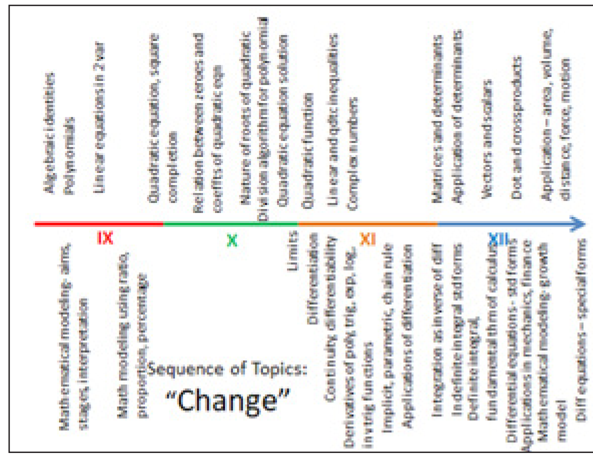
1. Number Systems
2. Algebra
3. Geometry
4. Mensuration
5. Coordinate Geometry
6. Trigonometry
7. Sequences and Series
8. Set Theory, Relations and Functions
9. Calculus
10. Vectors
11. Statistics & Probability
12. Commercial Math
13. Linear Programming
14. Mathematical Proof
15. Mathematical Modelling

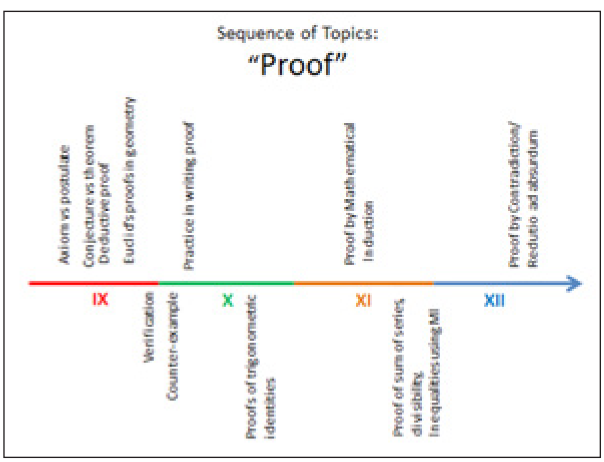
Syllabus comparison is the natural first step in developing a Math curriculum for gifted learners. An initial objective could well be to expand the learner's current curricular goals to include those adopted by other boards and systems. This would help the learner to achieve diverse learning goals in the same topics as well as expand the scope of the topics themselves.

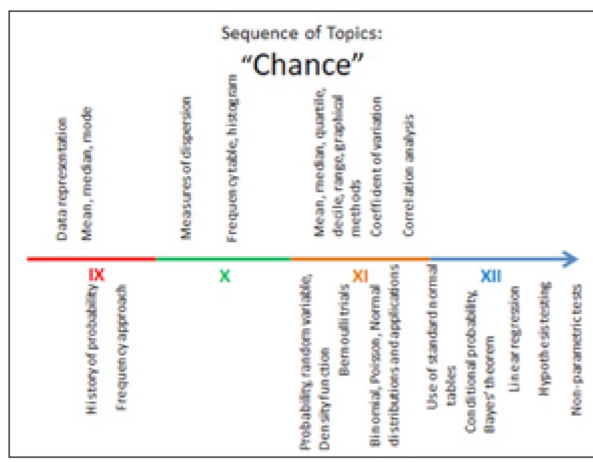
For an initial comparison, the course syllabus used by different examination boards, were collated. The topics are arranged approximately year-wise. We use a classification of the content of mathematics into 5 compartments: Counting, Proof, Space, Change and Chance.

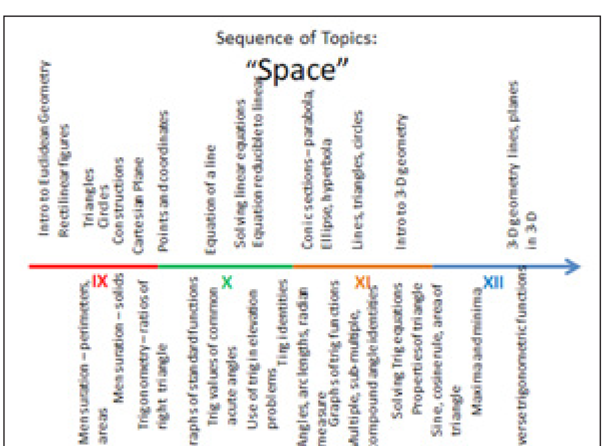
In the images shown alongside, the reader is expected to follow the syllabus topics from left to right. The continuity of the topics from Class 9 to Class 12, under the 5 thematic groups of Counting, Proof, Space, Change and Chance is mapped in these images.











A cursory glance at the above images makes one thing clear about the mainstream curriculum: it gives relatively little importance to topics that are grouped under the themes "Proof" and "Chance"; conversely, topics in Geometry/ Trigonometry and Calculus dominate the mathematics syllabus at the high school level. Some of this relative emphasis may change going forward, given the developments in science and technology.

The sequencing of syllabus topics from Class 9 to Class 12 is further elaborated in the form of spread-sheet extracts, in the Appendix. This mapping may be a useful reference resource for teachers.

The curricular structure and flow depicted above takes into account the broad pattern of math curriculum prescribed by the following Boards:

- ISC (and ICSE)
- CBSE
- AICE (Cambridge IGCSE & A-Levels)
- IB (International Baccalaureate and Diploma)

However, there are significant differences between specific Board requirements and preferences. The depth and treatment of the topics in the different Boards does vary substantially, as also the weightage accorded to formal theory vs intuition and practical application. Further, the Boards have significant differences in the modularization and optionality of the curricular contents. For example, the AICE Board completely modularizes Statistics and Applied Mathematics (Mechanics) and offers a choice between the two at the 12th grade. The AICE Board also modularizes the advanced mathematical content at the 12th grade and offers a separate paper for students who are interested in studying mathematics further. The IB system offers a specific module in Discrete Mathematics and offers modules at different levels of Math difficulty. The State Boards in India have their own syllabi. However, a cursory comparison between the State Board requirements at the 12th standard level and the ISC exam requirements in Mathematics shows that the State Board more or less follows the same pattern as the ISC, with somewhat reduced requirements in terms of coverage.

Given the flow of topics from the IX to the XII grade in Mathematics, students with a particular aptitude for Math at any of these grades have typically two options for exploration:

1) Explore mathematics at the next grade level
2) Explore mathematics at the same grade level, but with a broader scope

Typically, the student may do both of the above. However, as students graduate through the high school levels, their explorations are more likely to be of the latter type.

ENRICHMENT SOURCES

Restricted access to resources for learning is one of the primary constraints of a typical school system. Given limited time and the requirements of the examination-oriented study programme, teachers are better able to achieve their goals by using highly structured teaching and learning material, such as standard textbooks. While the use of textbooks aids teaching as well as learning, from the standpoint of the gifted learner, it is constraint, which limits exploration.

Going beyond prescribed textbooks also places the educators in uncertain terrain where it may be difficult to navigate. Keeping up with the students' explorations may also prove to be a challenge. Thus, one of the main challenges in a gifted learning curriculum is the identification of suitable enrichment sources.

Gifted students are likely to find many sources of enriched learning, especially using online resources. However, as the experience of any student or educator will confirm, online explorations, even when focused on a single topic, tend to be divergent, and therefore must be focused sharply on a particular problem or problem type, to be most productive. Students would benefit from a curated set of resources, preferably resources for each topic they are expected to explore.

Alternatively, students may enrol into a community that regularly posts challenge problems and solutions. In this regard Math Stack Exchange (for more advanced students) and Brilliant.org are excellent resources.

Characteristics of suitable enrichment sources include the following:-

- The starting point / pre-requisites of the material are in the school curriculum at that grade level
- The material is developed by competent persons and therefore can be deemed to be 'reliable'
- The material is self-contained, at least to a large extent. That is, the material is treated in sufficient depth and answers at least some of the related questions. If this were not to be so, the learner would drift from one to the other resource without learning much.

Identifying online material of the above kind could well be a 'supertask', but the advent of online social networks has made some of the curation for the above requirements feasible. User-reviewed material within online academic and professional networks and interest-groups, are now commonly available. Some of these groups have successfully distinguished themselves via their strong internal processes of governance: self-review, oversight and feedback. This includes restrictions on who can post or comment on material. Public rating of portals and contributors helps in achieving transparency in this regard. Thus, several websites and bloggers have distinguished themselves and have achieved a strong worldwide following in their specific areas of expertise. Educators must follow these trends if they are to be able to assist gifted learners.

In many cases, the learners are already following many of these trend-setting practices and accessing high-quality online content. Educators need to be able to stay on the same page.

A Gifted Learning Math programme cannot, on the other hand, be restricted to extant curricular frameworks. For example, history of science, which contains important context-setting information for various mathematical tools, discoveries and practices, is not taught at the school level at all. Exposure to the history of mathematics is an important area of intervention in this context.

In a similar vein, familiarity with the methods and practices of Indian mathematicians gives deep insight into the nature of mathematics. Indian students in particular will find such exposure highly empowering. There is a wealth of anecdotal information that suggests that many of the brilliant mathematical and scientific minds from India were influenced to a greater or lesser extent by the ingenious traditional computation methods they picked up at an early age. Since this tradition is a source of mathematical competence and excellence, it needs to be fully leveraged.

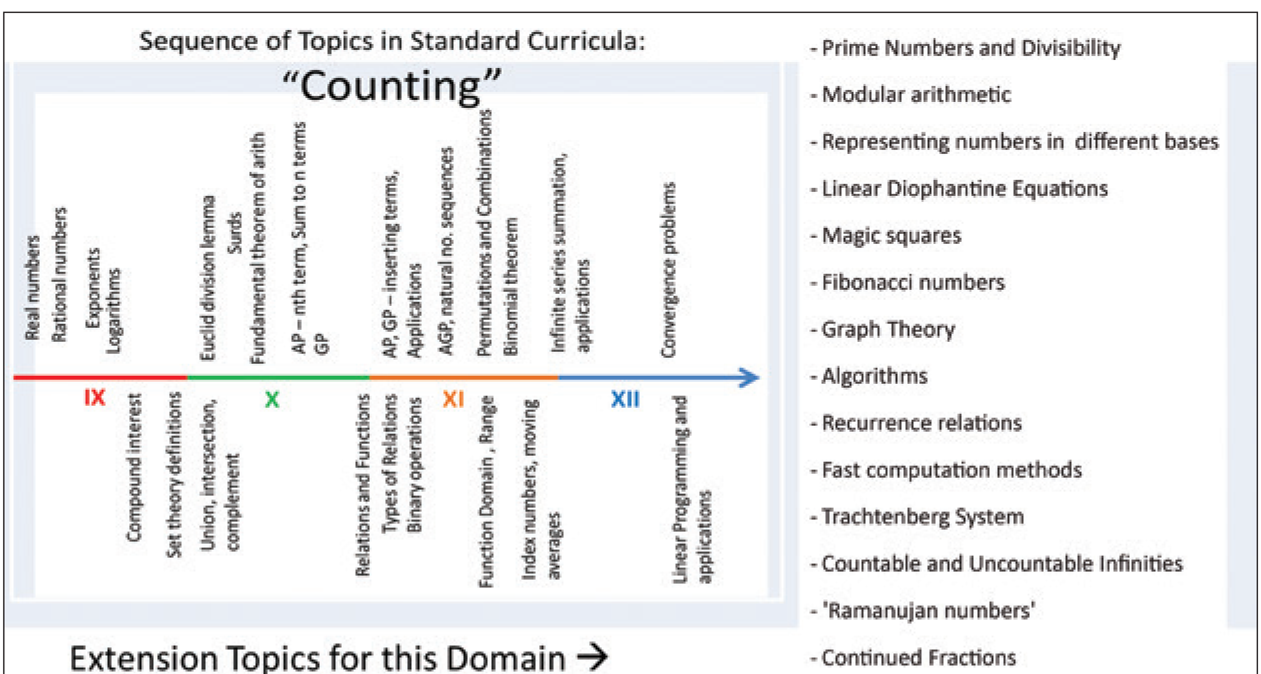
Gifted students should of course be exposed to puzzles and challenge problems as well. A large part of the learning process takes place in the context of trying to solve a challenging problem or puzzle, rather than from proving results using predictable methods or predictably using such proved results.

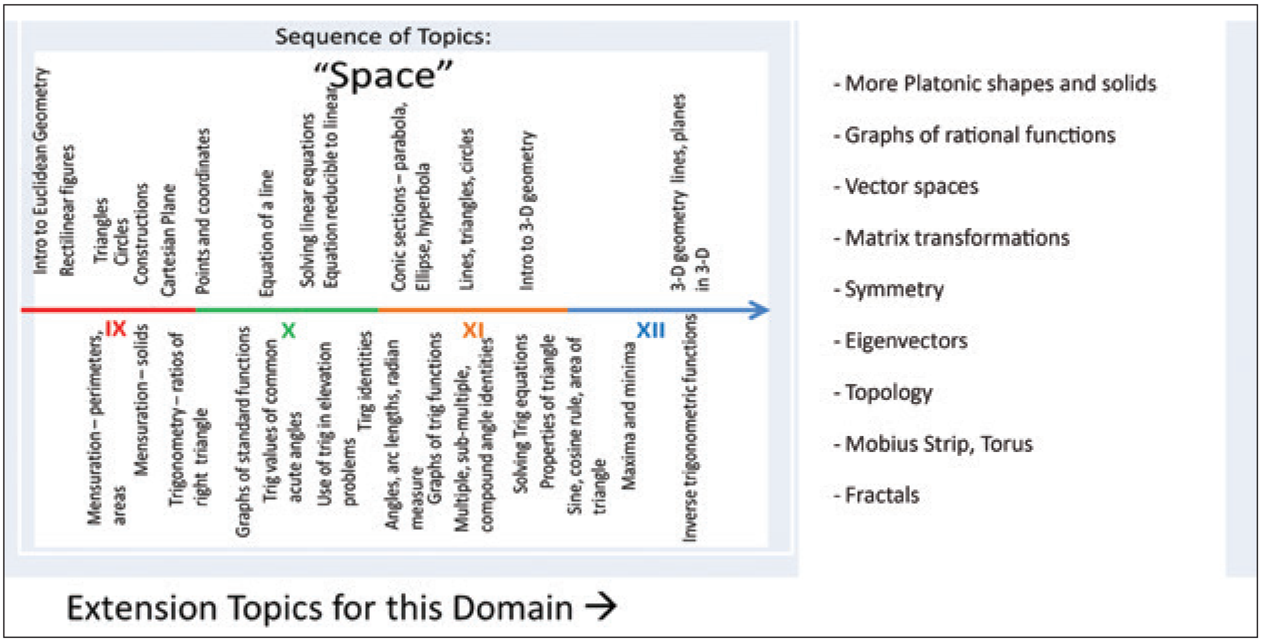
Resources for a Gifted Learning programme in Mathematics may incorporate all or several of the following principles:

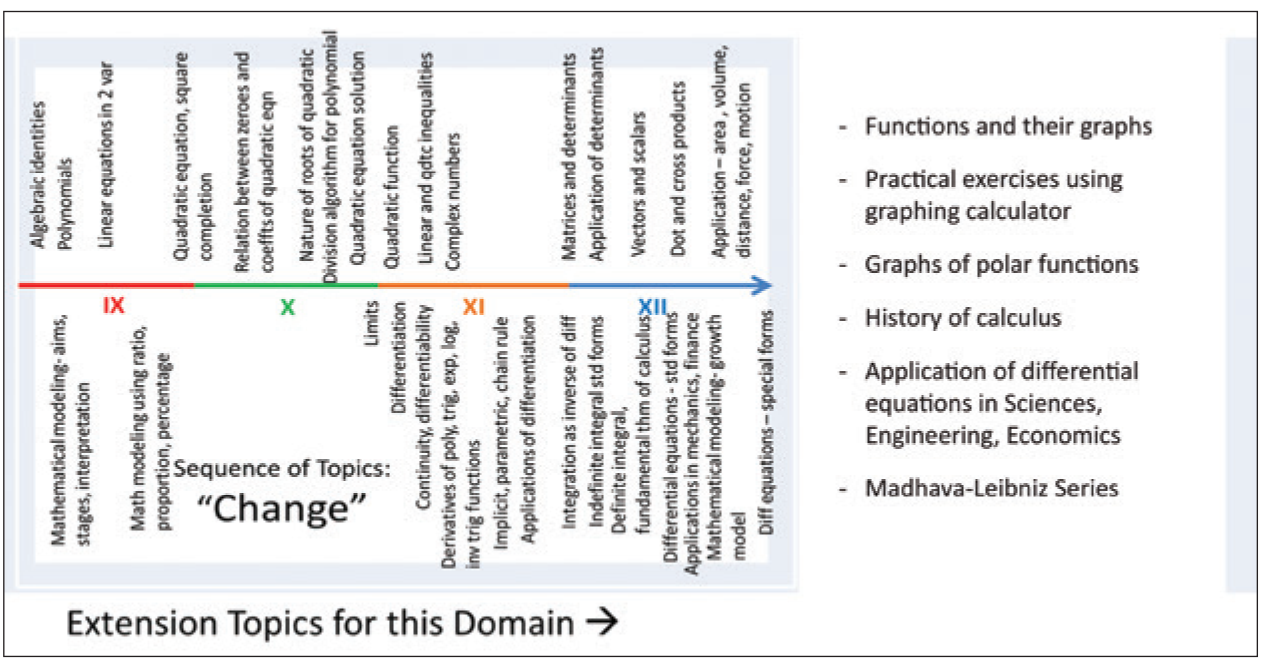
1	Use of pan-Indian syllabi and text-books
2	Use of international syllabi and materials
3	Online math lessons -- e.g. Khan academy https://www.khanacademy.org/
4	Online forums: e.g. Math Stack Exchange https://math.stackexchange.com/; or Brilliant.org https://brilliant.org/
5	Use of intuition. E.g. BetterExplained https://betterexplained.com/
6	Contextual learning. E.g. how coordinate geometry is used in astronomy; how matrices and determinants are used in computer graphics/ animations
7	Application-oriented study. Solving problems from applications or practical situation. E.g. problems in derivatives from commerce or mechanics
8	Fun learning and exploration; e.g. exploring fractals or polar coordinates
9	Use of math software: Desmos, Geogebra, Mathematica, R
10	Studying history and origins of mathematics
11	Student Projects
12	Essays, problems, solutions from publications like At Right Angles azimpremjifoundation.org/content/right-angles
13	Math Olympiad problems
14	Vedic or Indian Mathematics sources
15	Biographies of great mathematicians including Indian mathematicians, statisticians and scientists

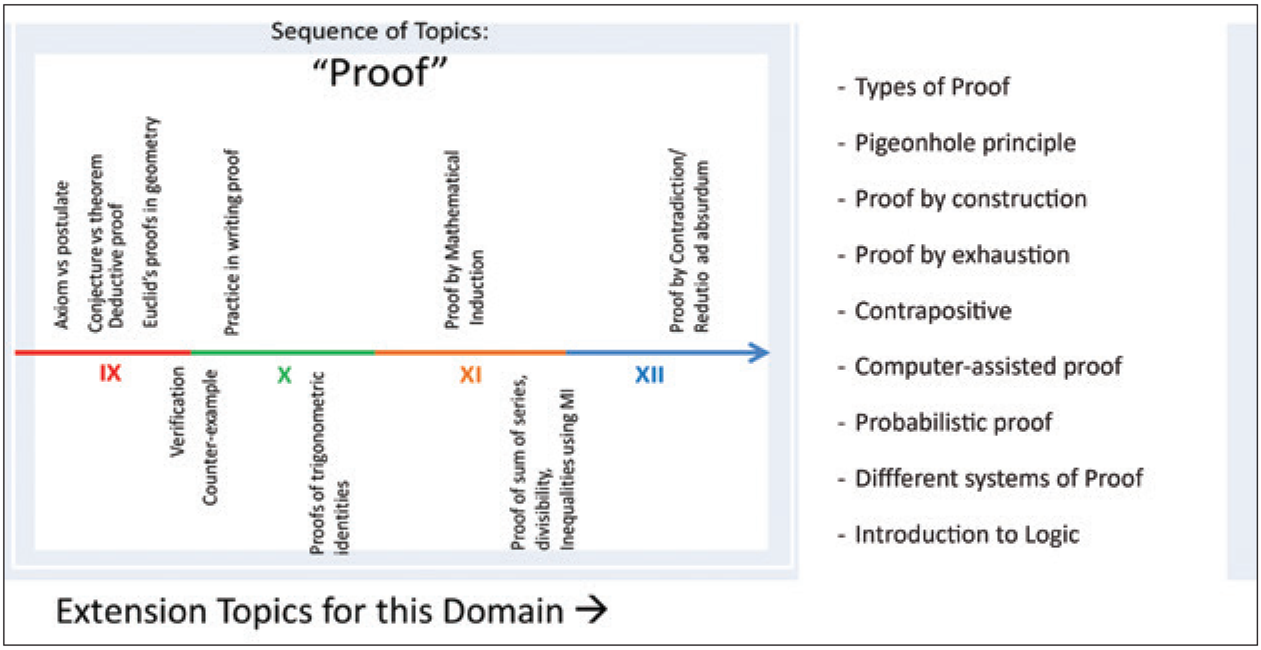
While some topics are better explored in one or other resource, in general, it is the learner and the educator who must identify the right material for a particular exploration. The gifted learning programme is by nature exploratory, rather than expository. It would be counter to its spirit to prescribe specific resources for specific topics, other than at a high level.

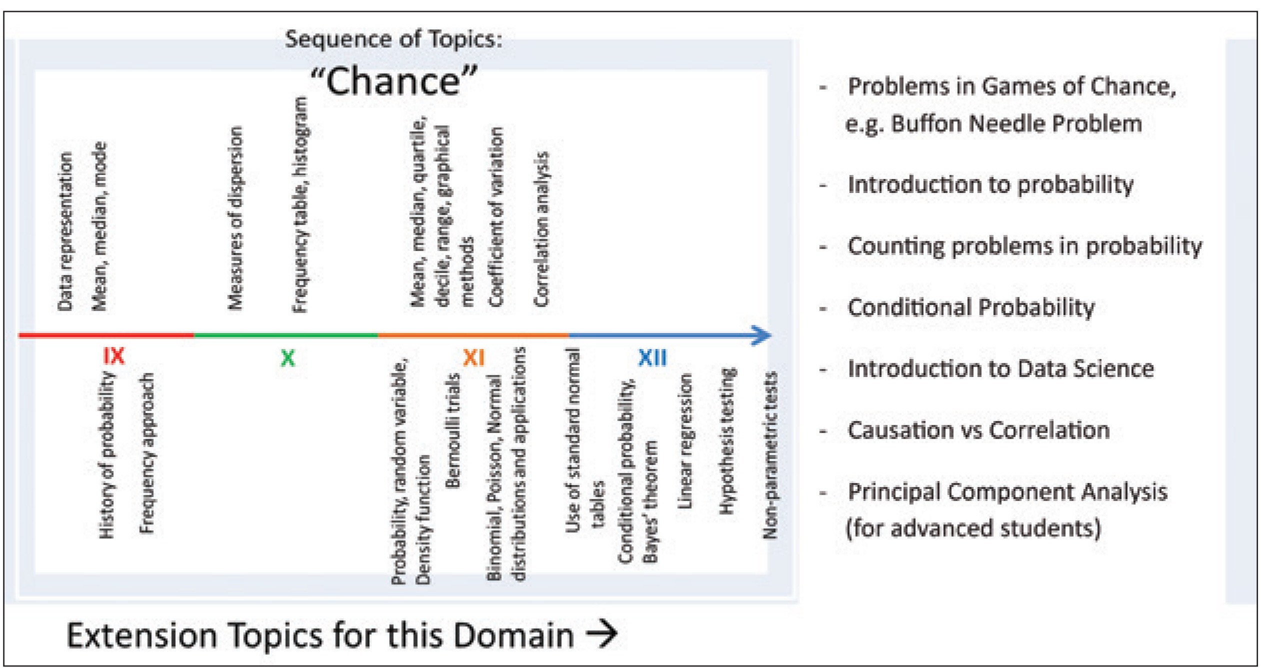
Suggestions for extension areas are indicated in more specific terms below. However, these are by no means a prescription, and should not be seen as constraints:




Sequence of Topics:
"Space"
Intro to Euclidean Geometry
Rectilinear figures
Triangles
Circles
Constructions
Cartesian Plane
Points and coordinates
Equation of a line
Solving linear equations
Equation reducible to linear
Conic sections – parabola, Ellipse, hyperbola
Lines, triangles, circles
Intro to 3-D geometry
3-D geometry lines, planes in 3-D
IX
X
XI
XII
Mensuration – perimeters, areas
Mensuration – solids
Trigonometry – ratios of right triangle
Graphs of standard functions
Trig values of common acute angles
Use of trig in elevation problems
Trig identities
Angles, arc lengths, radian measure
Graphs of trig functions
Multiple, sub-multiple, compound angle identities
Solving Trig equations
Properties of triangle
Sine, cosine rule, area of triangle
Maxima and minima
Inverse trigonometric functions
- More Platonic shapes and solids
- Graphs of rational functions
- Vector spaces
- Matrix transformations
- Symmetry
- Eigenvectors
- Topology
- Mobius Strip, Torus
- Fractals
Extension Topics for this Domain →


Algebraic identities
Polynomials
Linear equations in 2 var
Quadratic equation, square completion
Relation between zeroes and coeffts of quadratic eqn
Nature of roots of quadratic
Division algorithm for polynomial
Quadratic equation solution
Quadratic function
Linear and qdtc inequalities
Complex numbers
Matrices and determinants
Application of determinants
Vectors and scalars
Dot and cross products
Application – area, volume, distance, force, motion
IX
X
XI
XII
Mathematical modeling - aims, stages, interpretation
Math modeling using ratio, proportion, percentage
Sequence of Topics:
"Change"
Limits
Differentiation
Continuity, differentiability
Derivatives of poly, trig, exp, log, inv trig functions
Implicit, parametric, chain rule
Applications of differentiation
Integration as inverse of diff
Indefinite integral std forms
Definite integral, fundamental thm of calculus
Differential equations - std forms
Applications in mechanics, finance
Mathematical modeling - growth model
Diff equations – special forms
- Functions and their graphs
- Practical exercises using graphing calculator
- Graphs of polar functions
- History of calculus
- Application of differential equations in Sciences, Engineering, Economics
- Madhava-Leibniz Series
Extension Topics for this Domain →


Sequence of Topics:
"Proof"
Axiom vs postulate
Conjecture vs theorem
Deductive proof
Euclid's proofs in geometry
Practice in writing proof
Proof by Mathematical Induction
Proof by Contradiction/ Redutio ad absurdum
IX
X
XI
XII
Verification
Counter-example
Proofs of trigonometric identities
Proof of sum of series, divisibility, Inequalities using MI
- Types of Proof
- Pigeonhole principle
- Proof by construction
- Proof by exhaustion
- Contrapositive
- Computer-assisted proof
- Probabilistic proof
- Diffferent systems of Proof
- Introduction to Logic
Extension Topics for this Domain →


Sequence of Topics:
"Chance"
Data representation
Mean, median, mode
Measures of dispersion
Frequency table, histogram
Mean, median, quartile, decile, range, graphical methods
Coefficient of variation
Correlation analysis
IX
X
XI
XII
History of probability
Frequency approach
Probability, random variable, Density function
Bernoulli trials
Binomial, Poisson, Normal distributions and applications
Use of standard normal tables
Conditional probability, Bayes' theorem
Linear regression
Hypothesis testing
Non-parametric tests
- Problems in Games of Chance, e.g. Buffon Needle Problem
- Introduction to probability
- Counting problems in probability
- Conditional Probability
- Introduction to Data Science
- Causation vs Correlation
- Principal Component Analysis (for advanced students)
Extension Topics for this Domain →

For each of the topics mentioned in the above, material may be sourced from the internet or elsewhere. A bibliography is appended to this report, with a list of specific resources, which contain a wealth of usable material which the instructor may adapt to the needs of the learner.

ADDITIONAL CONTENTS (Beyond Standard Curricula)

Discrete Mathematics, also called Finite Mathematics, is an area of growing importance, particularly due to the explosive developments in computer science, computational methods and data science. Discrete mathematics refers to the study of mathematical structures which are not continuous (like points on a line) but are separate, like whole numbers, or vertices of a cube. Discrete mathematics includes topics like Logic, Set Theory, Number Theory, Combinatorics, Graph Theory, Probability, Operations Research, Decision Theory and also Topology, Algebra and Geometry. It is a fascinating and wide area of study with plentiful applications, as can be inferred from the above.

The International Baccalaureate (IB) programme offers a separate optional course in Discrete Mathematics that is not covered in any of the other curricula. In view of the tremendous application of this topic across Computer Science, Statistics, Economics, Industrial Engineering, Philosophy, etc. it is considered to be of particular interest to mathematically curious students. The standard textbook (Paul Fannon, 2013) is recommended to students and teachers.

In this regard, selected papers from the journal titled Resonance (Vishwambhar Pati, 2001) is also recommended.

The scope of discrete mathematics is large. The following material is recommended for a Gifted Learning Programme in the senior school year (Std XII and beyond):

- Methods of Proof
- Prime Numbers
- Representation of Integers in different bases
- Linear Diophantine equations
- Modular arithmetic/ Fermat's little theorem
- Graph theory
- Algorithms on Graphs (e.g. traveling salesman problem)
- Recurrence relations

In addition to the above, the following content is recommended:

1. Computation: Math in the computer lab. Enhance math learning through computer programming. An example of the same is the program: ContextFreeArt, listed in the bibliography. Incorporating math and science in the computer curriculum, for example, by writing programs to find prime numbers between two given numbers, generating random numbers, doing a Monte Carlo simulation, etc. is an excellent way to explore mathematics computationally. This method is already being successfully used in the Advanced Learning Centres of the NIAS Gifted Learning Programme.

2. Recreational Mathematics: Mathematics as fun/ art. Appreciation of the beauty and fun aspect of mathematics is very important as it motivates students to do math just for the fun of it. Puzzles, games, finding patterns, etc. would fall into this category. Sudoku and Rubik's cube problems are examples.

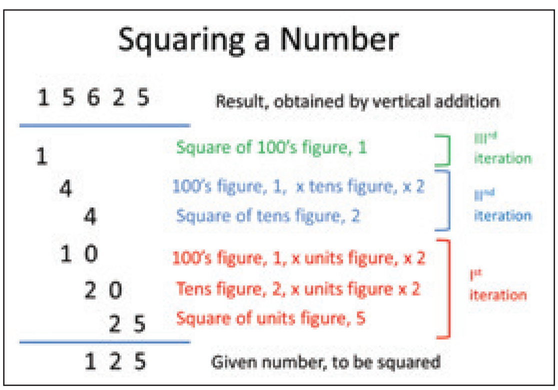
   A great collection of puzzles as well as the

UKTV Math programme "School of Hard Sums" can be accessed through this link:

https://dave.uktv.co.uk/dara-o-briain-school-of-hard-sums/daras-picture-puzzles/daras-logic-puzzles/

3. Speed Math: Doing arithmetic mentally is not something we focus on in schools any more. However, speed math is a special skill, which imparts insight into numbers. Vedic math and the Trachtenberg system are relevant here.

   In both systems, numbers are broken down in the decimal place value system and are handled in terms of the component parts



   For example suppose we have to square a number, say, 125. See the solved example. The rule used for solving is called a 'sutra' and has been documented, along with at least 15 other rules for speed math. Other (and faster) rules are also available for squaring a number

   Many books on the Vedic Math calculations are available in the market. The Trachtenberg system shares some commonalities but is said to be not the same.

   See: https://trachtenbergspeedmath.com/

4. Connecting mathematics to various disciplines as below:

   a. Mathematics and physics - non-euclidean geometry in relativity, classical and quantum world have different logic. Read this for an introduction to non-euclidean geometry: https://sites.math.washington.edu/~king/coursedir/m445w05/notes/non-euclidean.pdf

   b. Mathematics and biology - complex systems, chaos, fractals, cognitive neuroscience (fractals are discussed elsewhere in this document)
   A highly interesting lesson plan and discussion on sand piles as fractals may be found here:
   https://mathbabe.org/2012/07/19/hcssim-workshop-day-15-3/

   c. Mathematics and economics - game theory
   A rigorous treatment is provided, via examples, here: https://www.maa.org/sites/default/files/pdf/ebooks/GTE_sample.pdf
   Game theory expert Presh Walwalkar's videos are also a great resource:
   https://www.youtube.com/watch?v=zEeXnK11N04&index=2&list=PLDZcGqoKA84G4Ey-YdMEMqL8ssw6QSGu-

   d. Mathematics and music - patterns, rhythms
   A beginning in this direction can be made here: https://en.wikipedia.org/wiki/Music_and_mathematics

   e. Mathematics and pictures - origami/escher's paintings – geometry
   Large numbers of origami DIY videos

are available online. A start could be made with this simple activity to create a paper ‘diamond’: https://www.youtube.com/watch?v=BEKVE6jo0Jc
Students may read up the inspiring story of M Escher, and also attempt to copy his methods with some simple grid structures and create their own art, as a way of better appreciating the Escher paintings, rather than simply looking at them. A start may be made here:
http://im-possible.info/english/articles/escher_math/escher_math.html

f. Mathematics and philosophy - notions of truth, beauty, provability, formal systems, undecidability, paradoxes, logic
Brilliant.org (listed in bibliography) provides an excellent overview of this:
https://brilliant.org/wiki/introduction-to-paradoxes/

g. Mathematics and engineering/technology
Students and instructors may like to browse through this magazine for some inputs in this area:
https://plus.maths.org/content/ingenious-constructing-our-lives

h. Mathematics and computer science - cryptography, algorithms, graphs, computability
Brilliant.org provides many wiki pages on the topic of cryptography. Here’s one:
https://brilliant.org/wiki/public-key-cryptography/

i. Mathematics and sociology - social networks, sampling design, quantitative vs qualitative
AP Statistics provides a global pre-university level statistics curriculum. Students will benefit from reviewing the material here:
http://stattrek.com/tutorials/ap-statistics-tutorial.aspx

An introduction to sociology statistics is also available here:
https://www.thoughtco.com/introduction-to-statistics-3026701

j. Mathematics and humanities - archaeology, geography, mathematics as language, culture in mathematics, traditions and schools of mathematics
A list of engaging readings under the heading Mathematics and Language:
https://www.americanscientist.org/article/the-new-language-of-mathematics
https://www.quora.com/Is-math-a-language
http://pages.uoregon.edu/moursund/Math/language.htm

5. Mathematics Olympiad Problems:

Math Olympiad Problems merit separate mention. It is said (Gelca, 1956) that a good Olympiad problem will “capture in miniatoure the process of creating mathematics”. The INMO (Indian National Math Olympiad) (National Board for Higher Mathematics) activity was undertaken by NBHM from 1986 onwards and is currently run in collaboration with the Homi Bhabha Centre for Science Education, Mumbai. One main purpose of this activity is to support mathematical talent among high school students in the country.

The areas covered are:
- number systems,
- arithmetic of integers
- geometry
- quadratic equations and expressions
- trigonometry

- co-ordinate geometry
- systems of linear equations
- permutations and combinations
- factorisation of polynomials
- inequalities
- elementary combinatorics
- probability theory
- number theory
- infinite series
- complex numbers
- elementary graph theory

The syllabus does not include calculus and statistics. The typical areas for problems are: number theory, geometry, algebra and combinatorics. The syllabus is in a sense spread over class IX to class XII levels, but the problems under each topic are of an exceptionally high level in difficulty and sophistication. The difficulty level increases from RMO (Regional) to INMO (National) to IMO (International). NHBM recommends the following two books as reference material:

1. Mathematics Olympiad Primer, by V.Krishnamurthy, C.R.Pranesachar, K.N. Ranganathan and B.J. Venkatachala (Interline Publishing Pvt. Ltd., Bangalore)

2. Challenge and Thrill of Pre-College Mathematics, by V.Krishnamurthy, C.R.Pranesachar, K.N.Ranganathan and B.J.Venkatachala (New Age International Publishers, New Delhi - 1996).

Math Olympiad Challenges (Gelca, 1956) is a classic work that provides an excellent collection of challenge problems arranged under the headings:

1. Geometry and Trigonometry
2. Algebra and Analysis
3. Number Theory and Combinatorics

Problems are given along with an outline of solutions.

Here is a question that appeared in the INMO 2017:

Suppose n ≥ 0 is an integer and all the roots of $x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$ are integers. Find all possible values of α.

Model Solution:

Let u, v, w be the roots of $x^3 + \alpha x + 4 - (2 \times 2016^n) = 0$.

Then $u + v + w = 0$ and $uvw = -4 + (2 \times 2016^n)$.

Therefore we obtain $uv(u + v) = 4 - (2 \times 2016^n)$.

Suppose $n \geq 1$. Then we see that $uv(u + v) \equiv 4 \ (mod\ 2016^n)$.

Therefore $uv(u + v) \equiv 1 \ (mod\ 3)$ and $uv(u + v) \equiv 1 \ (mod\ 9)$.

This implies that $u \equiv 2 \ (mod\ 3)$ and $v \equiv 2 \ (mod\ 3)$.

This shows that $modulo$ 9 the pair (u, v) could be any one of the following: $(2,2), (2,5), (2,8), (5,2), (5,5), (5,8), (8,2), (8,5), (8,8)$.

In each case it is easy to check that $uv(u + v)\ 6 \equiv\ 4 \ (mod\ 9)$.

Hence $n = 0$ and $uv(u + v) = 2$. It follows that $(u, v) = (1,1), (1, -2)\ or\ (-2, 1)$.

Thus $\alpha = uv + vw + wu = uv - (u + v)\ 2 = -3$ for every pair (u, v).

Past Question Papers and Solutions are available on a website of TIFR (TIFR). This may be a very useful resource for the GLP-Math. Teachers and students may wish to jointly work through the solutions. It should be borne in mind that the problems are generally tough and will take time and efforts to work through.

5. Problems from the Indian mathematical tradition of Ganita

   Mathematically gifted Indian students ought to be exposed to the rich tradition of Indian Ganita.

   One problem is listed in the quiz section of brilliant.org: https://brilliant.org/problems/problem-from-lilavati-written-by-bhaskaracharya/

   A few other problems and references to the history of mathematics in India are scattered through the examples listed in this document.

   In addition, one may download a pdf version of a book on the subject (Satyanarayana, 2015). One of the examples quoted is as follows:

   Give three numbers which are equal when added with their one half, fifth, and one-ninth parts, and leaves 60 as a reminder, when each is diminished by those parts of the other two,.

Solution: Interpret the above correctly as follows.

Let x, y, z be the three numbers. Then, we are given that:

$$x + \frac{x}{2} = y + \frac{y}{5} = z + \frac{z}{9} \ (Equation\ 1)$$

and

$$x - \frac{y}{5} - \frac{z}{9} = 60 \ (Equation\ 2)$$
$$y - \frac{x}{2} - \frac{z}{9} = 60 \ (Equation\ 3)$$
$$z - \frac{x}{2} - \frac{y}{5} = 60 \ (Equation\ 4)$$

The trick is to equate the first equation to a parameter, say w. Then you can write x, y and z in terms of w.

The rest of the problem is left for the reader to solve.

Answer: $x = 100, y = 125, z = 135$

The above is just one example of ingenious algebraic problems set in Lilavati (Patwardhan, 2001). Students with an aptitude for this kind of work may be encouraged to access the full set of problems and attempt them.

5

Documentation of Selected Content Sources

Detailed content scoping

Grading of scope and associated problems

Lists or links to lists of problems

Model using Geogebra, Mathematica or other software

In this section, we provide a collection of sample lessons which may provide a practical and useful guide to gifted learners and their instructors. The topics for these lessons are not deliberately chosen, as any choice has to be arbitrary. Each topic in the syllabus can be extended in several directions. In the earlier classes, extension into the syllabus for the next level is one of the obvious choices. At higher levels, extension would necessarily enter into domains and topics that are usually taught at the University level.

In each of the sample lessons below, the following typical sequence of activities in expected:

1. The topic has already been covered in school/ in the programme the student is following
2. The student is eager to explore further. Educators may encourage the student's own explorations, typically using search phrases on the internet.
3. Educators may encourage students to browse through books and resources available at the local library.
4. Educators may prepare their own lessons and worksheets.

It is in the spirit of the last item above that the sample lessons of this section are presented.

The lessons presented are a selection from a large number of lessons created and delivered to students with mathematical aptitude over the past few years, in different contexts of socio-economic background, aspiration, linguistic sophistication (in English language), prior educational background, etc.

The materials are only to be considered as a sample. Educators must always apply their own judgment to the choice of materials to expose their students to.

Lesson and Evaluation Examples[6]

Trigonometry

Std. 9-10

Trigonometry is usually introduced at Class 8 with trigonometric rations based on a right triangle being presented in the form of definitions.

[6] The lesson and worksheet examples shown here are based on teaching and learning materials created by the author for use with various groups of students in the reference age-group, over the course of the past two years. The problems are sourced from a range of publicly available sources including standard textbooks and free-to-use web portals. A bibliography is appended at the end of this document.

Students may also be familiar with the mnemonic soh-cah-toa – which seems to be widely used in American and British schools.

However, the more meaningful way of introducing trigonometry is with reference to the circle. This is also the way trigonometry evolved, in the study of celestial objects and their distance from the observer.

Learning Objectives:

1. To define sine, cosine and tangent of an angle in the context of a circle and of a right triangle
2. Define and use radian measure
3. Convert radian measure to degrees and vice versa
4. Understand the relationship (analogousness) between arc of a circle and the angle subtended at the centre
5. Define polar coordinates in a simple and intuitive manner
6. Recognize that the y-coordinate is $rSin\theta$ and x-coordinate is $rCos\theta$
7. Learn the trig ratios of standard angles
8. Solve problems using the relationship between the trig ratios
9. Define the reciprocals of the three trig ratios
10. Establish the sign of the trig ratios in the 4 quadrants
11. Learn and use the ASTC mnemonic (or All Silver Tea Cups or equivalent)
12. Establish familiarity with the cyclic nature of the trig functions and the nature of
13. Use the terms amplitude and frequency in describing the trig functions
14. Use Geogebra to plot trig functions and generate familiarity with the form
15. Establish the simple rules such as Sine rule, Cosine rule
16. Use sine rule to find missing angle or missing side of a triangle
17. Use cosine rule to find the missing angle
18. Make simple diagrams given information about trig ratios to depict a situation mathematically

History of Mathematics

It is useful to introduce gifted learners to ideas from the history of mathematics in relation to the topic of trigonometry. Teachers may research the topic "Aryabhata's sine tables" to gain some information about this.

$Sin\theta$ and $Cos\theta$ are defined in Indian Ganita texts as respectively *jya* and *kojya*. Etymologically, it is claimed that the words Sine and Co-sine, come from these words by the following series of transliterations: *Jya*, also called *Jiva* in Sanskrit, was transliterated as jb in Arabic (the vowels were left out) and pronounced as Jaib, which means pocket in Arabic. The Latin translation of pocket is Sinus. Thus we have Sinus and Co-sinus or sine and cosine!

Taking the time from sunrise in the horizon to midday when the sun is directly overhead, and dividing it into 24 parts, Aryabhata calculated the 24 values of $r \times \sin\theta$, as the height of the celestial object (sun) above the horizon. His values are found to be extremely accurate.

Aryabhata lived in the 5th century, therefore it is clear that the trigonometric ratios are very old.

Practice work:

Problems for practice work involving trig ratios of standard angles can be found in the textbooks for Std.8,9,10 and 11. A large number of such problems is recommended.

An interesting aid to memory (Aggarwal, 2014) for the standard values is as shown below:

Ratio\Angle	0^o	30^o	45^o	60^o	90^o
Sine	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{4}{4}}$
Cosine	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{0}{4}}$
Tan	$\sqrt{\frac{0}{4-0}}$	$\sqrt{\frac{1}{4-1}}$	$\sqrt{\frac{2}{4-2}}$	$\sqrt{\frac{3}{4-3}}$	$\sqrt{\frac{4}{4-4}}=\infty$

Worksheets of interesting problems may be prepared. Here is an example:

Sample Problem:

If θ is an acute angle and $\frac{Cos\theta+Sin\theta}{Cos\theta-Sin\theta}=\frac{1+\sqrt{3}}{1-\sqrt{3}}$, find the value of $2Sec^2\theta-3Cosec^2\theta$.

Solution:

Problems which present information in the form $\frac{a}{b}=\frac{c}{d}$ can often be simplified by first applying componendo-dividendo: $\frac{a+b}{a-b}=\frac{c+d}{c-d}$

In this case, $\frac{(Cos\theta+Sin\theta)+(Cos\theta-Sin\theta)}{(Cos\theta+Sin\theta)-(Cos\theta-Sin\theta)}=\frac{(1+\sqrt{3})+(1-\sqrt{3})}{(1+\sqrt{3})-(1-\sqrt{3})}$

Hence, $\frac{2Cos\theta}{2Sin\theta}=\frac{2}{2\sqrt{3}}$

=> $\frac{1}{tan\theta}=\frac{1}{\sqrt{3}}$

Cross multiplying, $tan\theta=\sqrt{3}$

From the standard values, we know that $tan60^\circ=\sqrt{3}$

Therefore $\theta=60^\circ$ fulfils the above equation (not that it is not the only value that fulfils the equation!)

Putting $\theta=60^\circ$ in the expression $2Sec^2\theta-3Cosec^2\theta$, we get $2Sec^2 60-3Cosec^2 60$.

Knowing that $Sec\theta=\frac{1}{Cos\theta}$ and $Cosec\theta=\frac{1}{Sin\theta}$ we can evaluate the expression as:
$2(2)^2-3\left(\frac{2}{\sqrt{3}}\right)^2=2\times4-3\times\frac{4}{3}=8-4=4.$

This is an example of how an apparently complicated expression can be quite easily evaluated with some manipulation of the trig ratios.

Resources:

NASA has a website through which rocket science is made simple and accessible to students. In a section on maximum altitude of the rocket, the use of trigonometry is demonstrated. In addition, rocket flight is modelled using vetor algebra. Further, a project exercise is provided for the student to create a tool for calculating the distance of distant objects. This site is recommended: https://spaceflightsystems.grc.nasa.gov/education/rocket/rkthowhi.html

Every high school textbook has an extensive section dedicated to trigonometry.

Apart from ML Aggarwal, mentioned earlier

(Aggarwal, 2014), all recommended CBSE, ICSE and State Board textbook for the classes 8,9,10 contain extensive amounts of practice problems. Exam guides for Class 10 are also a good source of hundreds of practice problems.

The GCSE course (UK equivalent of Class 10) offers a slightly different set of problems and would therefore be a good additional resource. Letts' GCSE Mathematics (Patmore, 2000)for example offers problems of the following kind:

Example: Use your calculator to make a table of the values of sin x correct to 2 decimal places (d.p.) for values of x from 0^o to 180^o to 10^o in steps of . Using these results, plot the graph of y=sin x for $0^o \leq x \leq 180^o$.

The graph so obtained is called the sine curve or sine wave and is used to model many natural phenomena, including sound and light waves, heart beats and other types of cyclic phenomena. Khan Academy provides an excellent introduction to Trigonometry:

https://www.khanacademy.org/math/trigonometry

Kalid Azad provides unique methods of intuition to understand and use trigonometry. In particular, his interpretation of sine, cosine and tan are very interesting and is recommended to each gifted learner and their instructor. Teachers may wish to go over the material along with their students:

https://betterexplained.com/articles/intuitive-trigonometry/

As usual, Brilliant.org has a great selection of problems in trigonometry and learners can set their difficulty level at basic, intermediate or advanced.

https://brilliant.org/wiki/trigonometry/

Geogebra has built-in trig functions. This is only one of several user-developed public pages on trigonometry. Students can also switch between degrees and radians to develop their grasp further:

https://www.geogebra.org/m/zctE8msW

Students should be encouraged to explore a wide range of resources on their own.

Evaluation can be done by assigning worksheets developed from the many resources mentioned above.

Sample Lesson: Quadratic Equations

Std 10-11

Learning Objectives:

1. Solve a given quadratic equation by factorization
2. Solve a quadratic equation by using the quadratic formula
3. Draw conclusions about the nature of the roots from the discriminant
4. Recognize the graph of a quadratic function
5. Gain familiarity with some applications of quadratic equations

Typical problems involving quadratic equations are the Compound interest problem, EMI problem and Life Insurance problem. The so-called "Rule of 72", whereby one can calculate the rate at which an amount must be invested in order that it may double in a given number of years or the number of years required to double an investment which is growing at a particular rate can be calculated respectively by dividing 'number of years'/72, or multiplying 'rate of interest'x72 – will be of interest to students. It must be pointed out that this thumb-rule only gives an approximate result.

History of Mathematics:
Quadratic equations may have first arisen in commercial mathematics. An interesting problem is noted in the works of Aryabhata (CE 499).

A sum of money is lent out, earning x units in a month.

The interest portion, is lent out at the same rate for a period of t months.

The total yield on the principal is known (A). Determine the rate of interest (x).

In this problem the money-lender has already decided how much he (or she) wishes to earn by way of profit on the money that is being lent. The money-lending model is itself interesting – after recovering the principal from the first loan, the interest portion is now loaned out. The interest rate is negotiable. So the moneylender picks the rate of interest that will yield the return desired!

This must be one of the earliest known applications of quadratic equations. Note that the unit of time in the below is 'month'. Interest rate mentioned is monthly rate; term is in months. Students may be asked to work on this problem of Aryabhata. Solution:

Principal	P
Rate	x/p
Term	t
Yield	$A=x+tx^2/P$

Note that the yield is modelled as a quadratic equation. A look-up table for the desired returns and associated period of the loan is given below:

Students may observe that A is fixed, P is also fixed and by solving the quadratic equation for different values of t, you get the different values of x. Here, the problem is: "If I have Rs.10000, at what interest rate should I lend it so that I can make, say Rs.1000 in 6 months?" It seems in this case, that the moneylender is quite free to pick the rate of interest that suits him/her!

It is assumed in the above that the student already knows how to solve a quadratic equation using the formula.

Standard reference material:
A Das Gupta and A Banerjee (Banerjee, 2010) provides an extensive list of good quality practice problems and applications in this topic. Of particular interest will be the practical problems. Here are some examples:

1. The product of two consecutive integers is 132. Frame an equation for the statement. What is the degree of the equation? (Ans: $x^2 + x - 132 = 0, 2$).

Money-lending Business Model	A	2400											
	P	10000											
		Term	1	2	2	4	5	6	7	8	9	10	11	12
		Rate of Int	20.0%	17.7%	16.2%	15.0%	14.1%	13.3%	12.7%	12.2%	11.7%	11.3%	10.9%	10.6%
	A	1200											
	P	10000											
		Term	1	2	2	4	5	6	7	8	9	10	11	12
		Rate of Int	10.8%	10.0%	9.4%	8.9%	8.4%	8.1%	7.8%	7.5%	7.3%	7.0%	6.8%	6.7%

2. Find two consecutive odd numbers whose product is 323. (Ans: 17, 19)
3. The age of a father is twice the square of the age of his son. Eight years hence, the age of the father will be 4 years more than 3 times the age of the son. Find their present ages. (Ans: 32, 4)
4. If the sum of the two smaller sides of a right angled triangle is 17 cm and the perimeter is 30 cm fin the area of the right angled triangle. (Ans: 30 cm^2)
5. The line segment AB is 8 cm in length. AB is pulled to P, such that $BP^2 = AB.AP$. Find the length BP. (Ans: $4(\sqrt{5}+1)cm$)
6. A trader buys x articles for a total cost of Rs.600.
   a. Write down the cost of one article in terms of x.
   b. If the cost per article were Rs.5 more, the number of articles that were bought for Rs.600 would have been 4 less. Write down the equation in x for the above situation and solve it to find x.
   (Ans: x=24)
7. Two pipes working together can fill a tank in 35 minutes. If the larger pipe alone can fill the tank in 24 minutes less than the time taken by the smaller pipe then find the time taken by each pipe working alone to fill the tank.
   (Ans: the larger pipe takes 60 minutes and the smaller takes 84 minutes)

The Miscellaneous Applications section of the book provides problems that are better-suited to the needs of gifted learners.

Additional Resources are listed below:
EMI formula and how to work it out in Excel can be learnt here:
http://www.quickermaths.com/how-to-calculate-emi/

The same author also provides a concise summary of how to use the "Rule of 72" here:
http://www.quickermaths.com/rule-of-72-estimation-of-compound-interest-and-time/

Examples of application of quadratic equations from physics and life insurance calculation can also be easily found.

Complex Numbers

Std 11-12

Complex numbers are introduced in Std XI or XII, after an understanding of the real number system has already been established. The lesson objectives for this level are as follows:-

1. To be able to describe a complex number and recognize that a real number is a special case of a complex number
2. Use the Argand plane to solve problems involving complex numbers geometrically
3. Acquire terminology such as modulus, argument and conjugate
4. Arithmetic operations involving complex numbers
5. Locus of a complex number (expression)
6. nth roots of a complex number
7. Applications of complex numbers

Lesson Plan
It may be a good idea to start with context on complex numbers. A readable and sufficiently detailed history, which will be of interest to our learners, is provided here: http://www.math.uri.edu/~merino/spring06/mth562/ShortHistoryComplexNumbers2006.pdf

The Wikipedia page on complex numbers may be used to familiarize students with geometric representation of complex numbers both on the Argand plane and in terms of polar coordinates: https://en.wikipedia.org/wiki/Complex_number Addition and multiplication of complex numbers can also be explained using the above Wikipedia page. Subsequently the solutions of polynomials of the form $z^n+1=0$, viz. the roots of unity can be explored as shown on the same page.

Sufficient practice will be required to master the concepts in this topic. A comprehensive worksheet on complex numbers is provided in the later pages of this document.

Lesson Plan: Vector Algebra

Std 12

Learning Objectives:

1. To understand and be able to apply the concept of vectors and scalars
2. To manipulate vectors by the operations of addition, subtraction, multiplication
3. To interpret vectors geometrically, divide a line segment in a given ratio, find the components of a vector in a perpendicular directions
4. Dot and cross products, computation using determinants, geometrical interpretation

The following resources shall be used:

Standard Textbooks (recommended textbooks for PU/ ICSE / CBSE) cover the definitions, notations and types of vectors, including the following concepts:

- Vectors as directed line segments
- Magnitude and direction of a vector
- Types – equal, unit, zero
- Position vector
- Components of a vector, direction ratios and direction cosines
- Vectors in 2 and 3 dimensions
- Unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ and writing a given vector in terms of these unit vectors
- Operations on vectors (sum, difference, scalar multiplication)
- Interpretation of the sum of vectors, difference of vectors, scalar multiple
- Section formula
- Dot product and its geometrical interpretation
- Cross product and its application in area of triangle, area of parallelogram, collinear vectors
- Scalar triple product – volume of parallelepiped, co-planarity

The textbooks to be used (PU, ICSE, CBSE) provide vast amount of practice material and depth of coverage. However, the concepts of vectors, position vectors, etc. are often not clear to the students. For this, we make use of additional resources. Students are expected to have already covered the above material in their regular school. Even if they have not covered it, the following material can be accessed and interspersed with the classroom teaching:

1. Vector basics can be easily learnt and practised through the following video of Khan Academy. The material includes built-in short quizzes of a basic level, which help lay a strong foundation of understanding.
   https://www.khanacademy.org/math/precalculus/vectors-precalc

2. The following video of Khan Academy is a part of the Physics course, but can be used to further clarify the meaning of vectors and to gain insight into practical application.
   https://www.khanacademy.org/science/physics/one-dimensional-motion/displacement-velocity-time/v/introduction-to-vectors-and-scalars

3. Vectors can also be taught in the context of linear algebra and vector spaces, but it is not advised to take it up at this stage. It

can however be taken up in the context of matrices.

4. Dot and cross products are likely to confuse students and therefore it is good to develop a strong intuition about what they really are. This is provided by yet another Khan Academy video:
https://www.khanacademy.org/math/linear-algebra/vectors-and-spaces/dot-cross-products/v/dot-and-cross-product-comparison-intuition

5. A deeper dive into vector cross product is provided by Better Explained (Kalid Azad). The below link also looks at computational solutions using Wolfram Alpha, which is a free software that students can access. Trying out the software will also help students to see how these problems can be solved on computers:
https://betterexplained.com/articles/cross-product/

6. If time permits, the student may be encouraged to read up more on vector calculus:
https://betterexplained.com/articles/category/math/vector-calculus/

7. Teachers may supplement the study material with the following introduction to vector calculus from brilliant.org. The in-built quizzes may be used to test understanding and application:
https://brilliant.org/wiki/vector-terminology/

Testing:

1. Assorted vector problems based on CBSE syllabus requirements are to be found here:
http://www.learncbse.in/ncert-exemplar-problems-class-12-mathematics-vector-algebra/

2. A good set of problems balancing the computational and formula application aspects with the interpretation and analysis aspects can be found in the various textbooks prescribed for Cambridge A-Levels. An example of such a text book is:
Mathematics – The Core Course for A-Level, L Bostock and S Chandler, ELBS-Stanley Thornes, London, 1981, pp. 526-532

Project Work:

Std 12

1. Fractals
In this project, students explore the features of fractal images by creating patterns using a software package. Creating tree-like and grass-like shapes, as well as spirals and other geometric structures which are self-similar and nested, students begin to grapple with the complexity and at the same time simplicity of fractal models. The software package is available on the following link, along with user instructions and how-to guides.
https://contextfreeart.org/
Students may read Benoit Mandelbrot's original work "The Fractal Geometry of Nature" alongside of the above. They would also search and compile lists of fractal shapes, including the work of artists like Escher.

2. Number Theory
A number of interesting problems for students at the Std IX-XII level, offering challenge and insight into number theory are offered by the following organization:
https://promys.org/MathCircle/Resources
Students may download question papers from the Promys Challenge of each year (they are not available through the year on the website). The problems are both fun to do as well as insightful and challenging.

6

SAMPLE WORKSHEETS

In the following section, a number of practice worksheets have been compiled on various topics. The worksheets may be administered on a stand-alone basis, or in conjunction with a study programme linked to the relevant topic.

Example 1

Topic: Fractions & Decimals – Key Concepts

Level: Std IX

Lesson Objectives: Review of fraction and decimal manipulation

1. Numbers in which one integer is written over another are called fractions. Example:

$$\frac{5}{11}, \frac{3}{3}, \frac{17}{6}$$

2. Most numbers we deal with in life are fractions. These are also called RATIONAL NUMBERS. When it is written as $\frac{a}{b}$ we call it a fraction. For example, in $4, 0.7, 0.\overline{2222}, 20\%$ and $\frac{3}{11}$, only $\frac{3}{11}$ is a fraction, but all of them are rational numbers, because they can be expressed as fractions as follows:

$$\frac{4}{1}, \frac{7}{10}, \frac{2}{9}, \frac{20}{100}$$

3. Numbers that cannot be expressed as fractions are called IRRATIONAL NUMBERS. π is an irrational number. So is $\sqrt[n]{a}$, unless $a = b^n$

4. A fraction is in its lowest terms if no single positive integer greater than 1 is a factor of both numerator and denominator. For example, $\frac{8}{15}$ is in its lowest terms but $\frac{8}{18}$ is not.

5. Every fraction can be reduced to its lowest terms by dividing both numerator and denominator by their Greatest Common Factor. For example,

$$\frac{8}{18} = \frac{4}{9}$$

as the GCD of 8 and 18 is 2.

6. Every fraction can be expressed as a decimal (or whole number) by dividing the numerator by the denominator.
NOTE: If a fraction is written in lowest terms and if the only prime factors of the denominator are 2 or 5, then the decimal will TERMINATE.
Example: If the denominator is 4, 5, 8, 10, 16, 25 or 40, then the fraction gives a terminating decimal:

$$\frac{2}{10} = 0.2; \frac{5}{16} = 0.3125; \frac{26}{25} = 1.04$$

If the fraction is in its lowest terms and the denominator has prime factors other than 2 or 5, then the decimal will REPEAT.
Example: Since 6, 7, 9, 12, and 22 all have prime factors other than 2 and 5, fractions with these denominators yield recurring decimals:

$$\frac{5}{6} = 0.8\overline{333}; \frac{7}{9} = 0.\overline{7777}$$

7. To compare $\frac{a}{b}$ and $\frac{c}{d}$, cross multiply. If

$$ad = bc, then\ \frac{a}{b} = \frac{c}{d}$$

$$ad > bc, then\ \frac{a}{b} > \frac{c}{d}$$

$$ad < bc, then\ \frac{a}{b} < \frac{c}{d}$$

Example: $\frac{2}{3} > \frac{5}{8}$ since $2 \times 8 = 16 > 15 = 5 \times 3$

8. To multiply two or more fractions, multiply their numerators and multiply their denominators. Example:

$$\frac{3}{5} \times \frac{4}{7} = \frac{12}{35}$$

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

9. To multiply a fraction by any other number, write that number as a fraction whose denominator is 1:

$$\frac{3}{5} \times 7 = \frac{3}{5} \times \frac{7}{1} = \frac{21}{5}$$

$$\frac{3}{4} \times \pi = \frac{3}{4} \times \frac{\pi}{1} = \frac{3\pi}{4}$$

10. Before multiplying fractions, reduce.

11. To find a fraction of a number, multiply the fraction with the number. Since Percent is a fraction whose denominator is 100, you also multiply to find % of a number. Example:
If $\frac{2}{7}$ of the 840 students of a school are new and 30% of the new students play musical instruments, how many of the new students play musical instruments?
Answer: There are $\frac{2}{7} \times 840 = 240$ new students. Of these, $\frac{30}{100} \times 240 = 72$ play musical instruments.

12. To divide a number by a fraction, multiply that number with the reciprocal of the fraction.
Example: $20 \div \frac{2}{3} = \frac{20}{1} \times \frac{3}{2} = 30$

13. To add or subtract fractions with the same denominator, add or subtract the numerators and keep the denominator.

14. To add or subtract fractions with different denominators, first re-write the fractions as equivalent fractions with the same denominators. The easiest common denominator is the product of the two denominators. The best denominator is the LCM of the two denominators. Example:

$$\frac{1}{6} + \frac{3}{4} = \frac{2}{12} + \frac{9}{12} = \frac{11}{12}$$

$$\frac{2}{3a} + \frac{3}{2a} = \frac{4}{6a} + \frac{9}{6a} = \frac{13}{6a}$$

15. If $\frac{a}{b}$ is a fraction of a whole that satisfies some property, then $1 - \frac{a}{b}$ is the fraction of the whole that does not satisfy that same property.
Example: In a jar, ½ of the marbles are red, ¼ are white and 1/5 are blue. What fraction of the marbles are neither red, nor white, nor blue?
The red, white and blue marbles constitute

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{5} = \frac{10}{20} + \frac{5}{20} + \frac{4}{20} = \frac{19}{20}$$

part of the total. The balance is 1/20 (Answer).

Fractions & Decimals: Worksheet

1. Write 0.075 as a fraction in lowest terms.

2. Convert into decimal: $\frac{9}{11}, \frac{13}{55}, \frac{1}{7}$

3. Express $\frac{22}{7}$ correct to 4 decimal places.
You may know that this is one of the commonly used approximations for the irrational number π. Do you know of any other rational approximation for π?

4. Arrange in ascending order:

$$\frac{2}{3}, \frac{-4}{9}, \frac{-5}{8}, \frac{7}{12}$$

5. Without actual division, find which of the following are terminating:

$$\frac{11}{50}, \frac{48}{300}, \frac{27}{56}, \frac{514}{160}, \frac{987}{10500}$$

6. What fraction of 72 is $\frac{3}{5}$ of 80?
   a. $\frac{1}{3}$
   b. $\frac{2}{5}$
   c. $\frac{5}{9}$
   d. $\frac{3}{5}$
   e. $\frac{2}{3}$

7. If $0 < x < 1$, which of the following COULD be less than x?
   I. x^2 II. 10x% of x III. 1/ x
   a. I only
   b. II only
   c. III only
   d. I and II only
   e. I and III only

8. Which of the following numbers satisfies the inequality:

$$\frac{2}{3} < \frac{1}{x} < \frac{3}{4}$$

   a. $\frac{3}{5}$
   b. $\frac{7}{10}$
   c. $\frac{13}{10}$
   d. $\frac{10}{7}$
   e. $\frac{5}{3}$

9. If 5x=3 and 3y=5, what is the value of x/y?
   a. $\frac{9}{25}$
   b. $\frac{3}{5}$
   c. 1
   d. $\frac{5}{3}$
   e. $\frac{25}{9}$

10. On a test consisting of 80 questions, Raju answered 75% of the first 60 questions correctly. What percent of the other 20 questions did he need to answer correctly for her score on the entire test to be 80%?
   a. 85%
   b. 87.5%
   c. 90%
   d. 95%
   e. 100%

Example 2

Topic: Percentages

Level: Std IX

Lesson Objectives: Review of percentage problems

1. Percent means one-hundredth part.
   Example $23\% = \frac{23}{100} = 0.23$.

2. Remember percent is always with reference to some value.
   93% marks means 93 marks on 100.
   93% of 200 is 0.93×200=186

3. To convert % to decimal, drop the % sign and move the decimal two place to left.
   To convert % into fraction, drop the % sign and write the number over 100 and reduce to lower terms, if required.

4. To increase a number by r%, multiply it by $(1 + r\%)$
   To decrease a number by r%, multiply it by $(1 - r\%)$

5. If an initial quantity A increases r% per year, then the amount at the end of t years is given by $A(t) = A(1 + r\%)^t$

6. An increase of a% followed by an increase of b% always results in a larger increase than a single increase of (a+b)%.

7. A decrease of a%, followed by a decrease of b% always results in a smaller decrease than a single decrease of (a+b)%.

8. Percent increase of a quantity is $\frac{actual\ increase}{original\ amount} \times 100\%$

9. Percent decrease of a quanity is $\frac{actual\ decrease}{original\ amount} \times 100\%$

Percentages - Worksheet

The **percentage increase** of a quantity with reference to an original quantity is:

$$\frac{the\ actual\ increase}{the\ original\ amount} \times 100\%$$

The **percentage decrease** of a quantity with reference to an original quantity is:

$$\frac{the\ actual\ decrease}{the\ original\ amount} \times 100\%$$

Example 1 : From 1980 to 1990, the population of a town increased from 12,000 to 15,000. Since the actual increase was 3000, the percent increase in population was:

$$\frac{3000}{12000} \times 100\% = 25\%$$

Example 2: If, from 1980 to 1990 the population of a town increased from 12000 to 15000, and from 1990 to 2000 it increased by the same percent, then what was the percent increase in population from 1980 to 2000?

We already calculated the town's population increase was 25% from 1980 to 1990. If the population again increased by 25% from 1990 to 2000, that means it increased by:

$$\left(\frac{25}{100}\right) \times 15000 = 3750$$

Therefore, population in Year 2000 was . Therefore, from 1980 to 2000, the population increased by
which is $\frac{6750}{12000} \times 100\% = 56.25\%$ (Ans.)

EXERCISES:

TIP # 1: A % INCREASE IS NOT THE SAME AS A % OF AN ORIGINAL NUMBER.

1. The trading value of an antique stamp rose from Rs.100 to Rs.150. The current price is what % of the original price?
   a. 20%
   b. 25%
   c. 33 1/3 %
   d. 50%
   e. 150%

2. In the above problem, what was the % increase in price?
   a. 20%
   b. 25%
   c. 33 1/3 %
   d. 50%
   e. 150%

3. The trading value of an antique stamp declined from Rs.100 to Rs.50. What was the % decrease in price?
   a. 20%
   b. 25%
   c. 33 1/3 %
   d. 50%
   e. 150%

4. he trading value of an antique stamp went from Rs.100 to Rs.125. By what % would it have to be decreased to return it to its original price?
   a. 20%
   b. 25%
   c. 33 1/3 %
   d. 50%
   e. 150%

5. Over the course of a year, a certain printing house increased its output by 70%. At the same time, it decreased its total working hours by 20%. By what % did the printing house increase its output per hour?
   a. 50%
   b. 90%
   c. 112.5%
   d. 210%
   e. 212.5%

6. A gardener increased the length of his rectangular garden by increasing the length by 40% and width by 20%. The area of the new garden
   a. Has increased by 20%
   b. Has increased by 12%
   c. Has increased by 8%
   d. Is exactly the same as the old area
   e. Cannot be expressed in % without knowing the dimensions of original garden

TIP #2: YOU CANNOT ADD (OR SUBTRACT) THE % OF DIFFERENT WHOLES.

7. A passenger train increases its speed by 25% and then by 20%. What is its overall increase in speed?
   a. 25%
   b. 45%
   c. 50%
   d. 145%
   e. 150%

8. A discount of 20% on an order of goods followed by a discount of 10% amounts to
   a. The same as one 15% discount
   b. The same as one 22% discount
   c. The same as one 25% discount
   d. The same as one 28% discount
   e. The same as one 30% discount

9. The price of an item is decreased by 10% and then increased by 10%. The net effect on the price is
   a. No change
   b. Increase of 1%
   c. Decrease of 1%
   d. Decrease of 99%
   e. Depends on the actual price

10. The price of silver rose by 100% and then fell by 50%. Compared to the original price, the final price is
   a. 50% higher
   b. 25% higher
   c. The same
   d. 25% less
   e. 50% less

Tip #3. "RULE OF 72": Time for an investment to double with compounded interest

11. A business is growing at 10% per year. How many years will it take for the business to double in size?
   a. Less than 10 years
   b. Exactly 10 years
   c. Greater than 10 years
   d. Either less or greater than 10 years
   e. Depends on the size of the company

Assignment: Do some research on the Rule of 72. It says that if an investment is increasing at a rate of r%, the number of years taken to double will be approximately equal to 72/r.

Example 3

Topic: Algebra basic concepts & Visualization of functions

Level: Std IX -X

Lesson Objectives: Basic algebra review, simplification and visualization

Algebra – Review

Warm Up:

1. **Find the value of:**

$$\frac{x^5 - 4y^{-1}z}{\sqrt{y^2 - 8xz}}$$

when x=2, y=3, z=-1.

Ans: $6\frac{2}{3}$

2. **Simplify:**

$$-(x-3) - [x - (3 - 2x)]$$

Ans: 6-4x

3. **Multiply and Simplify:**

$$(x - 2y)(3x^2 + 2xy + y^2)$$

List of Special Algebraic Products (best to memorize)

i. $\boldsymbol{a(u+v) = au + av}$
ii. $\boldsymbol{(u+v)(u-v) = u^2 - v^2}$
iii. $\boldsymbol{(u+v)^2 = u^2 + v^2 + 2uv}$
iv. $\boldsymbol{(u-v)^2 = u^2 + v^2 - 2uv}$
v. $\boldsymbol{(u+c)(u+d) = u^2 + (c+d)u + cd}$
vi. $\boldsymbol{(au+c)(bu+d) = abu^2 + (ad+bc)u + cd}$
vii. $\boldsymbol{u^3 + v^3 = (u+v)(u^2 + v^2 - uv)}$
viii. $\boldsymbol{u^3 - v^3 = (u-v)(u^2 + v^2 + uv)}$

Using the Formulas:

1. Find the product: $(5s - 2t)(5s + 2t)$
2. Multiply $(6x^2 + 4y)^2$
3. Multiply and simplify: $(2x + 3y)(5x + 4y)$
4. Multiply and simplify: $(0.2c + \sqrt{2}d)(0.3c - 2\sqrt{d})$

Factoring:

This is the reverse of multiplying out as we did above. This process is helpful in solving equations.

1. Identify terms with common factors: Example: $3x + 6, 12x^4 + 8x^3y + 4x^2y^2, (x+y)(a+b) + (x+y)(a-b), x^3 + 2x^2 - 3x - 6$

2. Difference of two squares: Example: $4s^2 - 9t^2, 81a^4 - b^4, (a-5)^2 - (a+5)^2$

3. Perfect squares: Example: $x^2 - 6x + 9, 16x^2 + 40xy^3 + 25y^6$

4. Trinomials that are not perfect squares: Example: Factor $x^2 - 6x + 8, 2x^2 - 11x - 6, 4x^2 + 19xy + 12y^2$ $(by\ factorization\ of\ coeff t\ of\ x^2\ combined\ with\ some\ trial\ and\ error)$

5. The sum and difference of two cubes: Example: $x^3 - 8, 27a^3 + 64b^3, 8c^3 - \frac{1}{125}d^6$

General situation: Usually, several of the above will apply to a given problem. It usually helps to factor out the common factors first. Then look for the above patterns.

Problems:

1. Factor $3x^6 - 192$
2. Factor $5x^4 - 5x^3 - 20x^3 + 20x$
3. Factorize $50x^7y^2 - 80x^4y^3 + 32xy^4$

Rational Expressions:
An algebraic expression that is the quotient of two other algebraic expressions.

Example: $\frac{2}{3}, \frac{x^2-2x-3}{x^2-3x-4}, \frac{1-\frac{1}{x}}{x-1}$

What if you substitute a number into an expression and the denominator becomes zero? Then we say that the expression (or function) is NOT DEFINED for that value of x. It has to be specified whenever you use the expression.

Example: $\frac{1}{x}, x \neq 0, is\ well\ defined$

Sample Problems:

1. Find the value of $\frac{1-\frac{1}{x}}{x-1}$, if $x=-2,-1,0,1,3,8.47$

Note from this example:
1. It is important to evaluate ALL the denominators to determine where the expression is defined.
2. This expression can be simplified to $\frac{1}{x}$ Also remember:
$$\frac{ac}{bc} = \frac{a}{b}\ only\ if\ c \neq 0$$

Problems:
1. Simplify the following expressions and determine where they are defined:

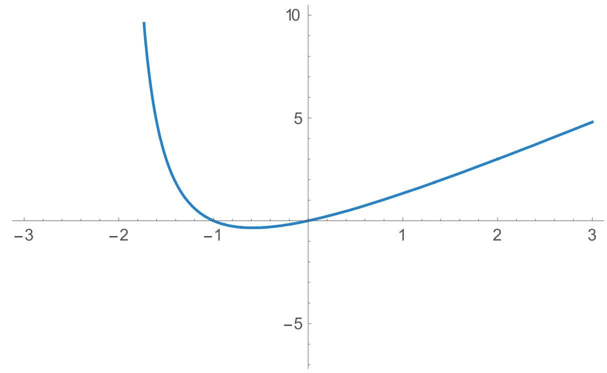
   a. $\frac{2x^3-5x-3}{x^2-9}$

   b. $\frac{x^2+x-2}{3-x-2x^2}$

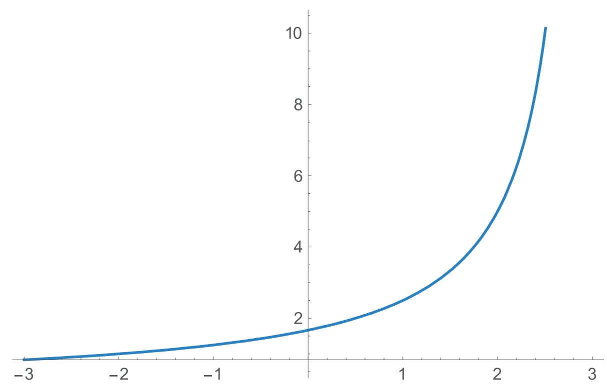
Hint: For a, $x \neq \pm 3$; For b, $x \neq 1$

More problems to solve in class:
Perform the operation, simplify and determine where the expression is defined:

1. $\frac{x^2-x-2}{6x-8} \div \frac{x^2-4}{12x^2-16x}$ Ans: $\frac{2x(x+1)}{x+2}, x \neq 0, \pm 2, \frac{4}{3}$

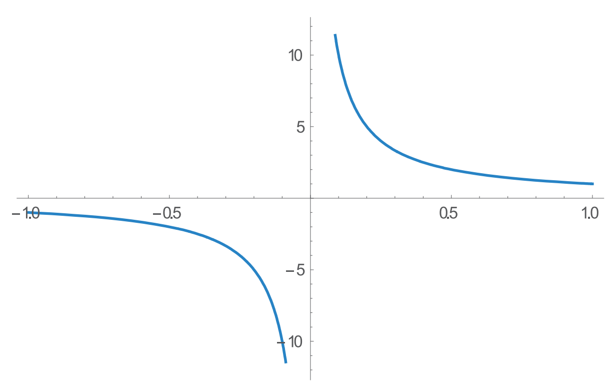


2. $\frac{x}{x+1} + \frac{2}{3-x} - \frac{x^2+3}{x^2-2x-3}$ Ans: $\frac{-5}{x-3}, x \neq 3, -1$

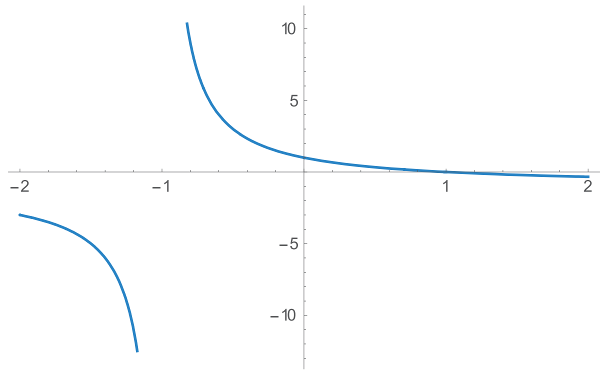


Simplify and determine where the expression is defined (Note that these rational expressions are called "compound fractions"):

3. $\frac{1-\frac{1}{x}}{x-1}$ Ans: $\frac{1}{x}, x \neq 0, 1$



4. $\dfrac{\frac{1-x}{x}-\frac{x}{1+x}}{\frac{1+x}{x}-\frac{x}{1-x}}$ Ans: $\frac{1-x}{1+x}, x \neq 0, \pm 1, \pm \frac{1}{\sqrt{2}}$



In each of the graphs above, study what is happening at the 'undefined' points. Notice that math packages don't always help you identify the 'undefined points'! You have to do your own analysis. Do not substitute models and packages howsoever popular, for your own analysis and interpretation.

Algebra Worksheet

(Simplify/ Factorize/ Rationalize)

1. Simplify:
   a. $y(y+2)+y-y(3y-2)$
   b. $(3s^2-4s^{-2})^2$
   c. $(2x-y)^2(2x+y)^2$
   d. $(3z-1)(2z-3)(3z+1)$

2. Factorize:
   a. $256-a^8$
   b. $24x^3-50x^2y+24xy^2$
   c. $28c^2-4c-9$
   d. x^3-3x^2-2x+6

3. Simplify:
   a. $\frac{(8m^{-3}n^{3/5})^{1/3}}{(27mn^{4/5})^{-1}}$
   b. $\sqrt[3]{125a^6b^{-3}}\sqrt[3]{0.001a^{-9}b^{12}}$
   c. $\frac{z-z^{-2}}{1-z^{-2}}$
   d. $\frac{3}{x-3}+\frac{2}{x^2-9}-\frac{4}{x+3}$
   e. $\frac{a^3-b^3}{a^2-b^2}$
   f. $\left(1-\frac{1}{x^2}\right)\Big/\left(\frac{1}{x}-1\right)$

4. Rationalize the numerator or denominator:
   a. $\sqrt{\frac{2}{11}}$
   b. $\frac{8}{\sqrt{x}-2}$
   c. $\frac{\sqrt{3+h}-\sqrt{3}}{h}$

Some Estimation Problems
(from Simon Singh – see bibliography)

1. If you took all the bowls of cereal that were on British breakfast tables this morning and tipped them onto the pitch at Wembley Stadium (before they were eaten), how deep would the pool of cereal be? (Imagine that that there was a wall built around the pitch, so that the wall and the pitch form a giant cereal dish.)

You will need to estimate a few things in order to decide the right answer:

   a. 1 m
   b. 10 m
   c. 100 m
   d. 1,000 m
   e. 10,000 m

2. If the whole world moved to Wales, and spread themselves out evenly, how much space would each person have on average?

a. 0.3 m^2 per person
b. 3 m^2 per person
c. 30 m^2 per person
d. 300 m^2 per person
e. 3000 m^2 per person

3. How many times could you say the alphabet in 24 hours? Assume that you will need to take appropriate breaks and will speak at a normal pace.

a. 100
b. 500
c. 5,000
d. 75,000
e. 750,000

Example 4

Topic: Ratio, Proportion, Rate

Level: Std IX or X

Lesson Objectives: Review of ratios and proportions

What is a Ratio?
A Ratio is fraction that compares two quantities which are measured in the same units.

Example: In the right triangle ΔABC, the length of the height $\overline{AC}$ is 6 cm and length of the side $\overline{BC}$ is 8 cm. Then we say that the ratio of AC to BC is 6 is to 8, written as , and it is just the fraction $\frac{6}{8}$.

The fraction may be converted to a decimal or a % as follows:-

$$AC : BC = 6 : 8 = \frac{6}{8} = \frac{3}{4} = 0.750 = 75\%$$

What this means is that there are an infinite number of triangles whose sides are in this ratio and ΔABC happens to be one of them. That is all.

Suppose ΔXYZ also has sides XZ and YZ in the ratio 6:8.

- Does this mean that XZ=6 and YZ=8?
- Is it possible that: XZ=24, YZ=32?

In fact there are infinitely many possibilities:

XZ	6	3	9	30	2.4	600, etc.
YZ	8	4	12	40	3.2	800, etc.

Question: What is the ratio of the Circumference to the diameter of a circle? Why? What does it mean?

Can there be a circle with a different ration of C:D?

What about right triangles? Can there be a right triangle with a different ratio of the 2 sides from the 3:4 given above?

Related concept: Geometric Scaling, 'Similarity' of geometric objects

Example 2: Can I describe the height: yield of coconut trees as a ratio? Discuss. Suggest some other measurements which can give meaningful ratios for comparison purposes.
Ratios are unit-free because the units in the numerator and denominator cancel out.
A fraction that compares two quantities measured in different units is called a <u>rate</u>.
Thus the yield of coconut trees of different heights, or ages, or the amount of water in different wells can be compared as rates, rather than ratios.

Note:
1. Ratios can always be written as fractions.
2. In any ratio problem, write x after each number and use the given information to solve for x.

Quick concept check:

1. What is the ratio of 5 hours to 3 hours?
2. What is the ratio of 5 days to 3 hours?

Example 1:
In a right triangle, the length of the shorter side to the longer side is 5:12. If the hypotenuse is 65, what is the perimeter?

Hints to Solution:

- Draw the right triangle and mark the sides 5x and 12x and the hypotenuse as 65.
- Use Pythagoras' theorem to set up an equation
- Solve for x (you have to solve a second-degree equation, but it is an easy one)

Example 2:
What is the degree measure of the largest angle of a quadrilateral, if the measures of the 4 angles are in the ratio 2:3:3:4?

Hints to Solution:

- Use the fact that the quadrilateral can be divided into two triangles and the sum of the angles of a triangle is 180 degrees
- Set up an equation by writing x next to each of the numbers of the angle ratio and summing the angles
- Solve this linear equation for x
- Calculate back the angle values

What is a Proportion?
A proportion is an equation that states that two ratios are equivalent. The following is a proportion:

$$\frac{6}{8} = \frac{3}{4}$$

Note that there is a fraction on both sides of the = sign. We say that the two quantities are proportionate.

Sometimes you will be presented with two ratios, and you will need to determine whether or not they are in proportion.

For example: if you need 6 cups of atta for 4 dozen puris, you will need 30 cups of atta for 20 dozen puris.

How do we know this? Because $\frac{6}{4}$ and $\frac{30}{20}$ are equivalent ratios. The ratio $\frac{30}{20}$ can be simplified back into $\frac{6}{4}$.

You can solve problems of proportions by cross-multiplying.

So, how many cups of atta are required for 10 dozen puris?

Solve: $\frac{6}{4} = \frac{?}{10} \Rightarrow ? = \frac{6\times10}{4} = \frac{60}{4} = 15$ cups (answer)

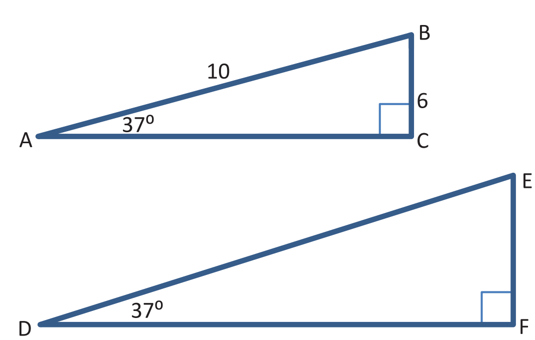
Example 1:

If $\frac{x+3}{19} = \frac{x+5}{20}, find\ x.$

Hints to Solution:

- Cross multiply the ratios and set up a linear equation in x
- Solve the linear equation for x
- x=35 (Answer)

Example 2: Refer to the diagram below. I copied triangle ABC and extended its sides while preserving the angles at A and C, and made the triangle DEF. Find the length of the side $\overline{EF}$.



Hints to Solution:

- The triangles are similar as the angles are preserved. This means that the ratio of the sides is preserved.
- In other words, the sides of ΔABC are proportionate to the sides of ΔDEF
- Thus you can set up an equation of ratios and solve for EF.
- EF=8.4 (Answer)

By the way, can we say that the sides of ΔABC are proportionate to the sides of ΔDEF?

'Proportionate to' is sometimes denoted by the symbol: ::

For example, how can I find the value of x here: 45:12 :: x:9?

Since they are in proportion, 45/12 = x/9. Hence 9 X 45 = x X 12. This gives x as 33.75.

What is Constant of Proportionality?

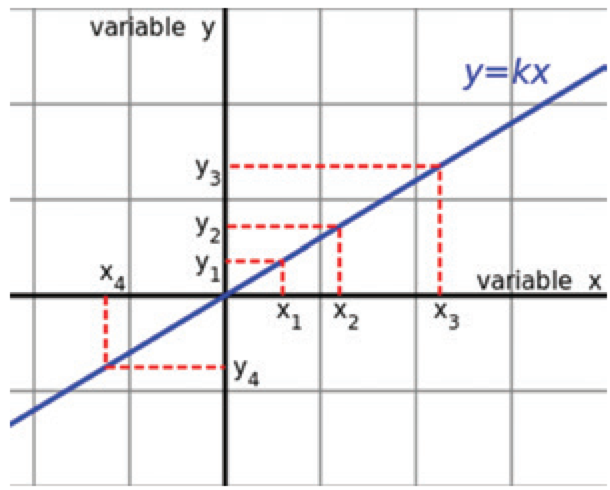
Two variables are proportional if a change in one is always accompanied by a change in the other, and if the changes are always related by use of a constant multiplier. The constant is called the coefficient of proportionality or proportionality constant.

- If one variable is always the product of the other and a constant, the two are said to be directly proportional. x and y are directly proportional if the ratio y/x is constant.

- If the product of the two variables is always a constant, the two are said to be inversely proportional. x and y are inversely proportional if the product xy is constant.

To express the statement "y is directly proportional to x" mathematically, we write an equation y = cx, where c is the proportionality constant. Symbolically, this is written as y ∝ x.

To express the statement "y is inversely proportional to x" mathematically, we write an equation y = c/x or xy=c. We can equivalently write "y is directly proportional to 1/x".



In Physics:

- If an object travels at a constant speed, then the distance traveled is directly proportional to the time spent traveling, with the speed being the constant of proportionality $s = kt, where\ k\ is\ constant\ equal\ to\ speed\ (u)$.

- The circumference of a circle is directly proportional to its diameter, with the constant of proportionality equal to π.

- On a map drawn to scale, the distance between any two points on the map is directly proportional to the distance between the two locations that the points represent, with the constant of proportionality being the scale of the map.'

- The force acting on a certain object due to gravity is directly proportional to the object's mass; the constant of proportionality between the mass and the force is known as gravitational acceleration.

$$F = Ma\ (where\ a = g, acceleration\ due\ to\ gravity;\ in\ this\ case\ F = weight)$$

In Biology & Chemistry: there are examples of rate constants – the initial rate of the reaction divided by the concentration of reactants. Look out for these in your Bio and Chem lessons.

In Economics: Let's say you are building airplanes of different sizes. The production cost of the airplane increases as the plane gets larger in size. Small size planes will have low cost, medium size will have medium cost, and large planes will have high cost. Let the size of the airplane equal to S and the cost equal to C. Notice that when S increases, C increases and when S decreases, so does C. If we can write $\frac{C}{S} = k$ then we can say that price is proportionate to size.

(This is a statistical problem and we are looking for perfect linear correlation about which you may learn later.)

What is Rate?

A rate is a fraction that compares two quantities measured in different units. Rates often use the word 'per', as in, 'price per metre', Kg/, calories/second, etc.

> You should set up rate problems just like ratio problems. Solve the proportions by cross-multiplying.
>
> Note that Rate problems are problems of direct variation or direct proportion.

Example 1: Jyoti can type 600 words in 15 minutes. Sudha can type twice as fast. How many words can Sudha type in 40 minutes?

Hints to Solution:

- Jyoti's rate is given, which is $\frac{600}{15}$. Sudha has twice the speed. So her rate should be: $\frac{1200}{15}$
- She is typing at that rate for 40 minutes. So you can find the number of words by multiplication of rate and time.
- This is equivalent to cross multiplying and solving the following equation: $\frac{1200}{15} = \frac{x}{40}$
- 3200 (Answer)

Example 2: If a apples cost b Paise, then how many apples can be bought for c Rupees?

Hint to solution:

- Note that the units should be made consistent. 100 paise = 1 Rupee
- Set up the proportionality equation: $\frac{Apples}{Paise} = \frac{a}{b} = \frac{x}{100c}$
- Cross-multiply and re-arrange making x the subject of the equation

Example 5

Topic: Exponents & Exponential Growth

Level: Std IX or X

Lesson Objectives: Review of exponentiation. Interpreting graphs. Introduction to e.

In the expression b^n, which is read as 'b raised to the power n', b is called the base and n the exponent.

Now, can b and n be:

1) Positive integers?
2) Negative integers?
3) Fractions?
4) Negative fractions?
5) Decimal numbers, positive or negative?
6) Zero?

For any number b and positive integer m,

1. if $b \neq 0$, then $b^0 = 1$
2. $b^1 = b$
3. If $m > 1$ *and m is an integer*, then $b^m = b \times b \times b \times \ldots (m\ times)$
4. $b^{-m} = \frac{1}{b^m}$

For example:
Take b=4 and m=0,1,2,-2,4, -4 and work out the values above.

Rules for Integral Exponents:
For any real numbers b and c and integers m and n:

1. $b^m \times b^n = b^{m+n}$
2. $\frac{b^m}{b^n} = b^{m-n}$
3. $(b^m)^n = b^{mn}$
4. $b^m \times c^m = (bc)^m$
5. $a^m = a^n \Rightarrow m = n$ (note that this works only on the same base)

Example: If $4^m \times 4^n = 4^{10}$ then what is the arithmetic mean of m and n?

Hint: Use the relevant formula to get the value of m+n.

Surds

Fractional indices are called surds. If a is any real number and n is a positive integer then x is called the nth root of a if and only if $x^n = a$. We can write $x = \sqrt[n]{a}$ *or* $a^{\frac{1}{n}}$.

If n is a rational number (Why?), and $n = \frac{p}{q}$ where p and q are integers with no common factors (Why?) and q>0 (Why?), then for any real number a:

$$a^n = a^{\frac{p}{q}} = \left(a^{\frac{1}{q}}\right)^p = (a^p)^{\frac{1}{q}}$$

Squares and Square Roots

Are you familiar with the following equations using squares? What are they?

1. $A = b^2$
2. $A = \pi r^2$
3. $a^2 + b^2 = c^2$
4. $x^2 - y^2 = (x + y)(x - y)$
5. $\sin^2\theta + \cos^2\theta = 1$
6. $x^2 + y^2 = r^2$

Exercise:

1. Write the first 15 natural numbers and their squares.
   These square numbers are called perfect squares.
2. Is there a square root for every positive number?
3. What does the graph of a square expression look like? (Quadratic)

For any positive numbers a and b:

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}\ and\ \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Exercise:

5. Write the first 10 powers of 2
6. A chessboard was filled with rice grains. 1 for the first square, 2 for the second, 4 for the third, etc. At which square does the number reach 1000 (or more)? When does it cross 100,000? If a grain of rice on average weighs 1/64 of a gram, what is the weight of the grains piled on the 20th square? When does the weight exceed one tonne?
7. If you draw a graph of the powers of 2, what will it look like? See: https://en.wikipedia.org/wiki/Wheat_and_chessboard_problem

8. Plot the integral powers of 1, 2, 3 on a sheet (i.e. $y = a^x$). Do the graphs intersect?

The problems with large numbers to be done using spreadsheet software.

Euler's Number: e

$$e \approx 2.718$$

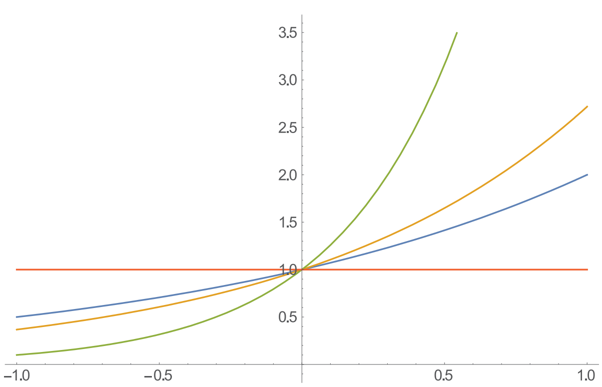
But what is e?

Compound interest, with compounding every instant $e = \left(1 + \frac{1}{n}\right)^n$, where n is very large. Try it out with various values of n.

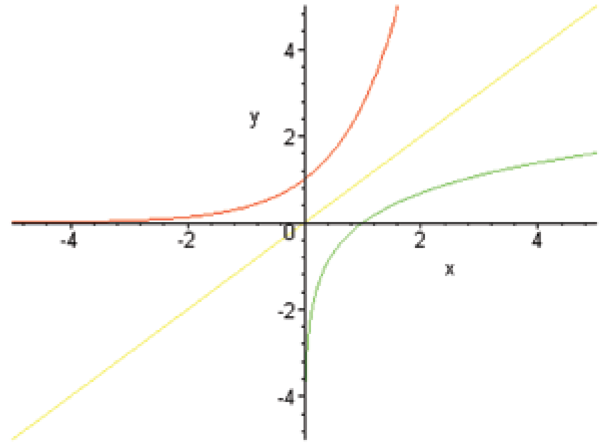
http://www.mathsisfun.com/numbers/e-eulers-number.html

http://www.popularmechanics.com/science/math/a24383/mathematical-constant-e/

In the below graph, can you identify: ?
$y = 2^x, y = e^x, y = 10^x$?



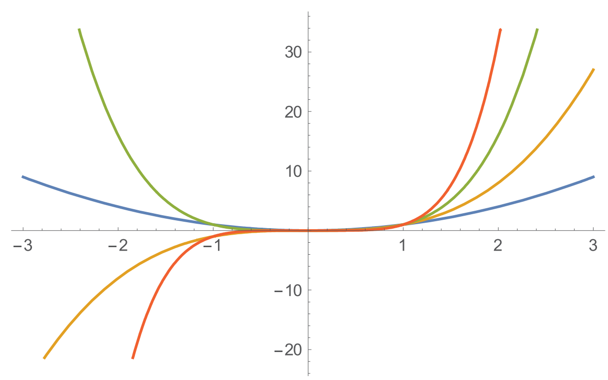
Check this out: The red line is $y = e^x$, the yellow line is y=x, the green line is $y = log_e x$ which is the inverse of $y = log_e x$. (More on logarithms later!) Do you see the reflection here?



In the following graph, can you identify:

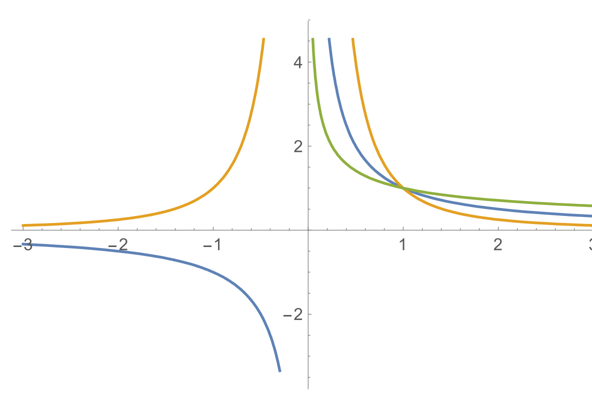
$$y = x^2, x^3, x^4, x^5$$

Write down your observations about the graphs of these functions. Which ones are symmetric?



In the following graphs, can you identity:

$$y = x^{-1}, x^{-2}, x^{-0.5}$$



Write down your observations about the graphs of these functions. Which ones are symmetric?

Exponents – Worksheet

Simplify if possible:-

1. $(-27)^{\frac{5}{3}}$
2. $4^{\frac{-5}{2}}$
3. $64^{\frac{4}{6}}$
4. $(-64)^{\frac{4}{6}}$
5. $(64^2)^{\frac{1}{3}}$
6. $(-64)^{\frac{5}{6}}$

Simplify these algebraic expressions:

7. $x^{\frac{1}{3}}(x^4)^{\frac{1}{6}}$
8. $\left(\frac{-27x^{-6}y^{\frac{3}{2}}}{z^{\frac{-4}{5}}w}\right)^{-\frac{1}{3}}$
9. $\sqrt[7]{x\sqrt{x^5}}$
10. $(\sqrt[3]{64})^{\frac{3}{2}}$
11. $\sqrt{7\sqrt[5]{7^3}}$
12. $\sqrt{12}\sqrt{75}$

Use your calculator to compute to 3 decimal places the following. Be careful while using calculator. Check if your calculator is doing the operations in the correct sequence. Check if your answer is making sense.

13. $15^{\frac{7}{11}}$
14. $3^{3/20}$
15. $8^{1/15}$

Example 6

Topic: Fascinating Geometric Ratios

Level: Std IX or higher

Lesson Objectives: Introduction to the Golden Ratio

Amazing Geometric Ratios: Paper sizing and Pretty Rectangles

Are all rectangles similar, in the way that all squares are similar and all circles and equilateral triangles are?

It turns out that rectangles are defined by the ratio of their sides. Rectangles with different side ratios look different, while those with the same ratio appear to be but magnifications of the same rectangle.

There are two types of problems we shall examine:

1. The A-series of paper sizes, and what the ratio of their sides has to do with their popularity
2. The Golden Ratio – what is unique and special about it

1. A-series of Paper Sizes

The A4 size is the usual size for photocopying paper. A5 is half the area of A4, A6 is half of A5 and so on. Likewise A3 is double the area of A4, A2 double of A3 and so on. These sizes are very efficient for factory production.

These doubles and halves of rectangular area may suggest something to the reader. If a rectangle has length l and width b, its area is lb. What would be the dimensions of a rectangle with area lb/2? If b>l/2, could the length of the new rectangle be b and width be l/2? Yes, of course.

Now, supposing $\frac{l}{b}=\sqrt{2}$, then $l=b\sqrt{2}$ and $lb=b^2\sqrt{2}$

Now, if the A4 is halved, then the new length is b and the new width is l/2, so that we can write:

$$l_1 = b,\ b_1 = \frac{l}{2}, l_1 b_1 = \frac{lb}{2} = \frac{b^2\sqrt{2}}{2} = \frac{b^2}{\sqrt{2}}$$

This is the size of the A5 sheet. If this sheet were to be further halved, we get:

$l_2 = b_1,\ b_2 = \frac{l_1}{2}, l_2 b_2 = \frac{l_1 b_1}{2} = \frac{b^2}{2\sqrt{2}}$ and so on.

Note that $\frac{l_n}{b_n} = \frac{l}{b} = \sqrt{2}$, for all values of n=1,2, 3... Thus, if the starting rectangle has sides in the ratio $\sqrt{2}$, the sides of the rectangles produced by halving the given rectangle preserves the ratio of the sides. This property is uniquely true for rectangles with sides in the ratio $\sqrt{2}$.

The inverse of this ratio is $\frac{1}{\sqrt{2}} = .707 = \frac{b}{l}$

This ratio is therefore memorable:

$\frac{b}{l} = \mathbf{0.707}$ for the A-series papers.

Activity:
Step 1: Measure the length and width of an A4 sheet to the nearest millimetre. Note it down

Step 2: Fold it into half along the longer side, and cut it along the fold. Measure the sides of the rectangular piece so obtained. Note it down. Mark the piece as A5.

Step 3: Repeat Step 2 until you reach the A10.

Step 4: Make a simple tabulation of the Paper size, length, width and the ratio of length to width.

Incidentally, do notice that the ratio is unit-free. The length and width are in mm.

Compare the ratios obtained. Do they hover in the region of 1.414?

Compute the area of the rectangles A4, A5, … A10. How are subsequent areas found from previous ones?

How are previous areas found from subsequent ones?

What is the area of an A0 sheet? Can you demonstrate that it is close to 1 sq m in your experiment?

2. Golden Ratio

The Golden Ratio refers to a different type of uniquely sized rectangle. The question here is: What are the dimensions of a rectangle, which when reduced by a length equal to its width, leaves another rectangle with sides in the same ratio as the original?

Let l and b be respectively the length and width of the starting rectangle.

We mark a length b along the length of the rectangle and cut out a bxb square. We are now left with a rectangle of size b x (l-b).

The requirement is that the ratios of the sides of this new rectangle be equal to the ratio of the sides of the original rectangle, which can be stated algebraically as:

$$\frac{l}{b} = \frac{b}{l - b}$$

That is, $b^2 + lb - l^2 = 0$

Solving this quadratic equation with $l = 1$, and taking only the positive root, we have:

$$b = \frac{-1 + \sqrt{1 + 4}}{2} = \frac{\sqrt{5} - 1}{2} \sim 0.618$$

Thus, the sides of the starting rectangle are in the ratio 1:0.618

If so, then the sides of the rectangle obtained by cutting out a square of side b from the original rectangle will also be in the same ratio.

And since the smaller rectangle has this property of ratio of the sides being 0.618, so will the rectangle obtained by cutting the square out of the smaller rectangle, and so on.

This unique ratio is called the **Golden Ratio** and **its value is approximately 0.618**.

This was regarded by the Greeks as the standard for mathematical 'beauty'.

An outstanding book on the topic, titled The Golden Ratio: The Story of Phi the World's Most Astonishing Number, published by Broadway Books, is available in pdf form on this link:

http://www.fm-lb.org/sites/default/files/Mario_Livio_-_The_Golden_Ratio.pdf

Example 7

Topic: Straight Lines and their Graphs

Level: Std IX or X

Lesson Objectives: Review of algebra and geometry of straight line

Linear functions are important because they model many phenomena but also because they are useful for approximating complex phenomena. The word linear comes from line. Graph of a linear function is a line.

Definition:
A linear function is a function (this term needs to be understood) whose rule is of the form:

$$f(x) = mx + b$$

where m and b are real numbers.

A fundamental property of a line is its slope.

Definition:
If a line *l* is not vertical and if $P(x_1,y_1)$ and $Q(x_2,y_2)$ are two distinct points on the line, then the slope of the line *l*, which is usually denoted by *m*, is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{change\ in\ y}{change\ in\ x}$$

Slope of a vertical line is not defined.

Exercise: Draw the straight line through (-1, 2) and (3, -1) and find its slope.

Ans: m=-3/4

Exercise: Sketch the line through the given points and find the slope of each line. You may sketch all the lines on the same graph.

a) (-1, 2), (4, 1)
b) (-1, 2), (1, 6)
c) (-1, 2), (3, 2)
d) (-1, 2), (-1,-1)

Note:
Suppose a line has slope m:

1. If m is $\begin{cases} positive \\ negative \end{cases}$ then the line goes $\begin{cases} upward \\ downward \end{cases}$
2. If m=0, the line is horizontal
3. If m is undefined, the line is vertical

Note: Does it matter which two points we use to calculate slope of a line? Why?/Why not? Explain by making a sketch.

Hint: Use the concept of similar triangles and the ratio of their corresponding sides.

Exercise: Find two other points on the line through (-1, -2), and sketch if the slope is

a) 3/2
b) -3

There are other ways to specify a line, using the points on it. Since $= \frac{y-y_1}{x-x_1}$ for all points (x,y) on the line, we can multiply both sides by $x - x_1$ and get the following:

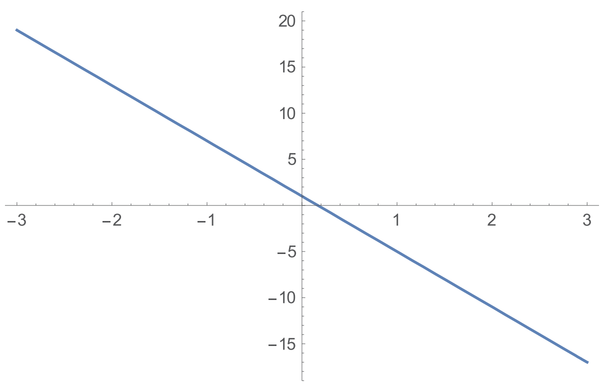
Definition: Point-Slope Form for Equation of Line

An equation for the line throught he point P=(x_1,y_1) and with slope m is:

$$y - y_1 = m(x - x_1)$$

Exercise: Find an equation of the line through (-1/3, 3) and (1,-5)

Ans: y+6x-1=0



Definition:
The y-intercept of a non-vertical straight line is the y-coordinate of the point where the line crosses the x-axis.

What is x-intercept?

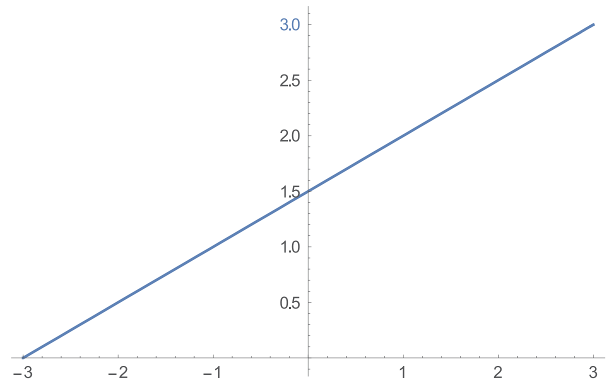
Definition: Slope-Intercept Form of Equation of Line
y=mx+b

Exercise: Put the equation 2x+3y+3=0 into slope intercept form, then graph it. Find the x-intercept from the graph.

Definition: Parallel Lines
Two lines are parallel if both have the same slope of are both vertical.

Exercise: Find the equation of a the line through the point (1, 2) and parallel to 2x-4y+5=0

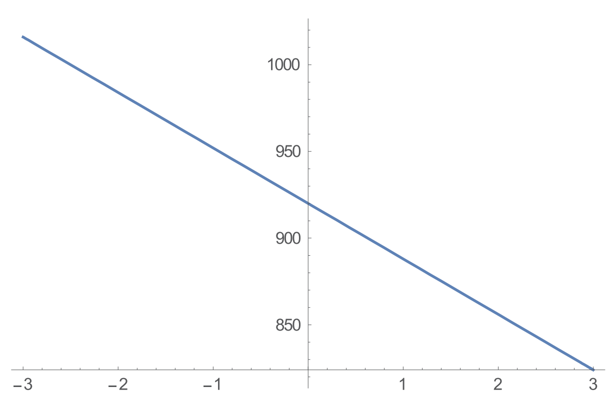
Answer: y=x/2 + 3/2



Exercise: If a projectile is shot straight up, its velocity v is a linear function of time t. Suppose a shell is shot straight up with an initial velocity of 920 feet per second and that after 5 seconds it is going up at a rate of 760 feet per second.

a) Find the equation which expresses v as a linear function of t
b) How long does it take for the shell to reach its highest point?

Hint: Do you have two points on the line?
Ans: a) v=-32t+920 (why negative slope?) b) 28.75 sec



Example 8

Topic: Compound Interest

Level: Std X or higher

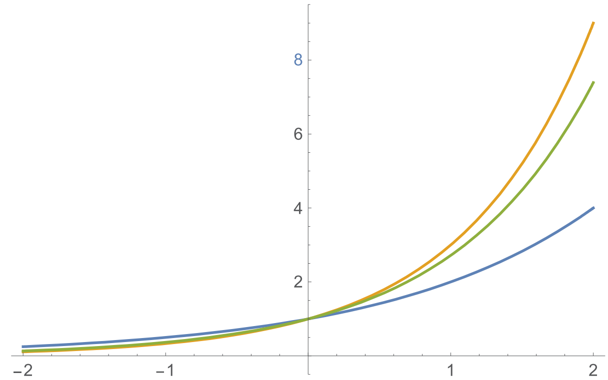
Lesson Objectives: What is compounding? What is the fastest compounding rate?

The compounding effect may take place in many situations – of animal populations declining in a forest, and of the reduced share of horses in the King's horse distribution formula for his sons – the problem of dividing 15 horses in the proportion 8:4:2:1, which was ingeniously solved by – we think – Tenali Raman!

Compounding is an effect that turns a simple linear phenomenon into a complex (or complicated) phenomenon, which can be depicted by a curve, not a straight line. A compounding effect cannot be depicted by a straight line in the usual Cartesian plane in which we draw graphs (i.e. unless you use a non-linear scale, e.g. logarithmic).

The expression compounding itself means, increasing the complexity, or aggravating something, or accelerating or decelerating it. Usually, it is some kind of geometric phenomenon, leading to values that rise or decrease 'geometrically', i.e. in powers of some multiplier.

We have seen graphs of exponential phenomena such as geometric growth by doubling, in the case of the chessboard and rice grains problem. We have looked at the graphs of $2^x, 3^x$ and e^x. They are shown below. You can easily say which is which.



Let us look at the compounding phenomenon closely, using the idea of compound interest.

Formula for Compound Interest

If P dollars (or rupees) are invested at a rate of interest of r per period, then the amount A that the investment is worth after n time periods is:

$$\boldsymbol{A = P(1 + \frac{r}{n})^n}$$

In this formula, it is important to note the concept of 'period' and 'rate of interest'.

If r% is offered for 1 year, and you are interested in the value of Amount at the end of 2 years, it is easy to compute:

$$A = P(1 + r)^2$$

For example, if Rs.1000 is invested at 8% p.a. for 2 years, then

$$P = 1000, \qquad r = .08, \qquad n = 2$$

$$A = 1000(1 + .08)^2 = 1166.4$$

If, however, you invested P at the rate of r% per annum, but wanted to compound every month, and find the value after 2 years, you would use the formula:

$$A = 1000\left(1 + \frac{.08}{12}\right)^{24} = 1172.89$$

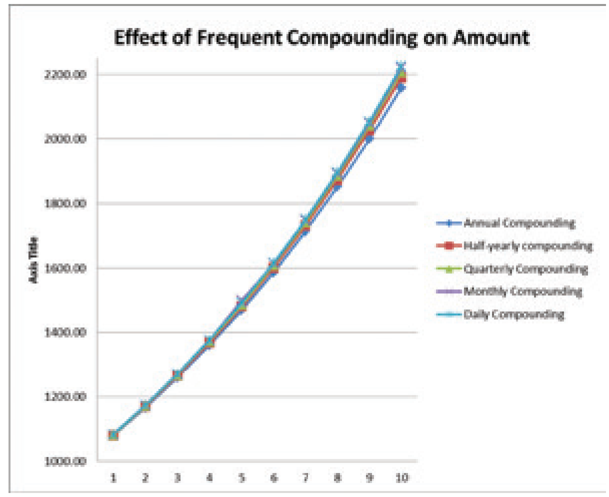
This increase in the value of Amount may make us think that it might be a good idea to compound more frequently. However, it turns out that the advantage of frequent compounding levels off after a certain point.

In the below example, some data are analysed in Excel in this regard. (Only partial data are shown for the interest obtained by daily compounding. The data appear on the next page.)

The summary of the way the principal grows in the different scenarios is below:

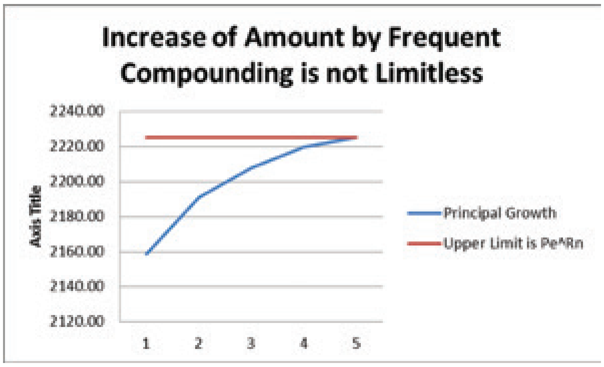
The interesting thing to note is how the advantage of frequent compounding levels off. The expression:

$\left(1+\frac{1}{n}\right)^n$ has been worked out for different (increasing) values of n, and found to approach the value 2.718, in fact an irrational value close to this number, which is designated *e* and is an interesting mathematical constant.



One of the ways in which this is interesting is that it tells us how far the advantage (or effect) of frequent compounding can go. Initially, for low values of n, the advantage is quite significant, but it keeps shrinking as n becomes larger and larger. In the data and graphs shown above, you can see that while half-yearly compounding is better than annual compounding (especially over a large number of years), it hardly makes a difference whether you compound the money daily or monthly. The presence of the expression:

Advantace of Compounding More Frequently Keeps Redudng						Gain				
	Annual Compound-ing	Half-yearly Compound-ing	Quarterly Compound-ing	Monthly Compound-ing	Daily Compound-ing	Annual Compound-ing	Half-yearly Compound-ing	Quarterly Compound-ing	Monthly Compound-ing	Daily Compound-ing
1	1080.00	1081.60	1082.43	1083.00	1083.28	80.0	81.6	82.4	83.0	83.3
2	1166.40	1169.86	1171.66	1172.89	1173.49	166.4	169.9	171.7	172.9	173.5
3	1259.71	1265.32	1268.24	1270.24	1271.22	259.7	265.3	268.2	270.2	271.2
4	1360.49	1368.57	1372.79	1375.67	1377.08	360.5	368.6	372.8	375.7	377.1
5	1469.33	1480.24	1485.95	1499.78	1491.76	469.3	480.2	485.9	499.8	491.8
6	1586.87	1601.03	1608.44	1613.50	1615.99	586.9	601.0	608.4	613.5	616.0
7	1713.82	1731.68	1741.02	1747.42	1750.57	713.8	731.7	741.0	747.4	750.6
8	1850.93	1872.98	1884.54	1892.46	1896.35	850.9	873.0	884.5	892.5	896.3
9	1999.00	2025.82	2039.89	2049.53	2054.27	999.0	1025.8	1039.9	1049.5	1054.3
10	2158.92	2191.12	2208.04	2219.64	2225.35	1158.9	1191.1	1208.0	1219.6	1225.3
	2225.3563	**2225.3563**	**2225.356**	**2225.356**	**2225.356**					
					1000					



$\left(1+\frac{r}{n}\right)^n$ in the formula for compound interest, is linked to this phenomenon. As n grows very large, in fact this expression becomes

Thus we have the following result:

Another Formula for Compounding:

If P Dollars (or Rupees) are invested at an annual rate of interest r, compounded continuously, then after n years, it is worth $\boldsymbol{A = P(e^r)^n = Pe^{rn}}$

You will see the link between the expression above and the shape of the compound interest graph.

What this means is that if you have a fixed sum of money P and a fixed interest rate r% p.a., and a fixed timeframe of investment n years, then the maximum amount that this can grow into, with as much (or frequent) compounding as possible, is Pe^{rn}.

It is easier to calculate the value of this number than the other formula and that is one good reason to know it. The other is that it shows the connection between a growth phenomenon growing with a certain growth rate, and e.

Example:
India's population (Jul 2017) is estimated at 1.34 billion.

Annual growth rate (current) is estimated at 1.33%

(Wolfram Mathematica gives this number)

Population after 13 years (in 2030)
$\leq 1.34 \times e^{(.0133\times 13)} = 1.593\ billion$

Other questions which could be asked are:

When will India's population be 3 billion (as per the above model. Not a real-life scenario!)?

OR

When Indian population reaches 3 billion, what will be China's population?

A concept linked to exponentiation and indices is that of logarithms. We do not discuss it in detail here. However, Wikipedia provides a good introduction:

https://en.wikipedia.org/wiki/Logarithm

In particular, students and teachers are advised to review the original Encyclopaedia Britannica entry on Logarithms (which is reproduced in the above website) and discuss it.

Example 9

Topic: Coordinate Geometry

Level: Std X or higher

Lesson Objectives: Review of distance formula. Reflection of a point.

I. Distance formula:

$$d(P,Q) = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

if P, Q are points (x_1, y_1), (x_2, y_2) on the plane.

$$d(P,Q) = |x_2 - x_1|$$

if P, Q are points on a line parallel to x-axis.

$$d(P,Q) = |y_2 - y_1|$$

if P, Q are points on a line parallel to y-axis.

Problems:

1. Find the distance between (3, -2) and (-1,-7)

2. Find the distance between (a,2) and (2,a). Sketch the graph of the line.

3. Use the distance formula to determine if these 3 points are collinear: P(-5, 3), Q(1,-1), R(4,-3)

Hint: If 3 points are not collinear, then they form a triangle; and in a triangle, the sum of 2 sides always exceeds the third side.

II. Mid-point formula

The mid-point of the segment between (x_1, y_1) and (x_2, y_2) is M $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

4. Find the mid-point of the segment between (-3,1) and (2,-5). Sketch.

Mixed Problems:

5. The points (1,2) and (1,6) are vertices of a square. Find the other two vertices. Sketch.

6. Choose suitable scales on the axes and plot and label the points:

   a. A (300,1), B(350,2), C(400,-1)

   b. A(0.1, -0.002), B(0.2, -0.004), C(-0.1, -0.005)

   c. A(-1, -5000), B(0, 2000), C(2, 3000)

7. Find the mid-points of:

   a. $(c^{1/2}, d^{1/2}), (-c^{1/2}, -d^{-1/2})$

   b. $(t, |t|), (-|t|, t)$

   c. $(a+b, a-b), (b-a, a+b)$

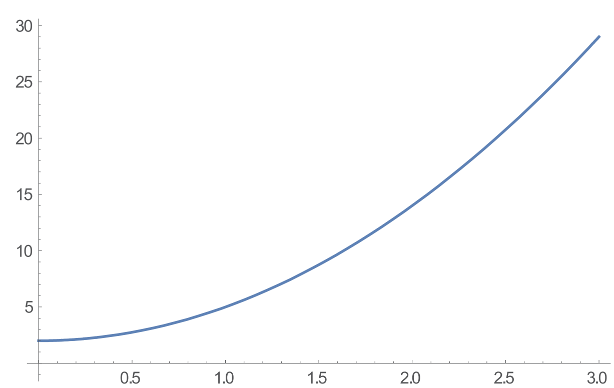
III. REFLECTION

We have already looked at the reflections of various graphs in the x-axis, in the line x=y and so on. Here are some simple rules to reflect a graph.

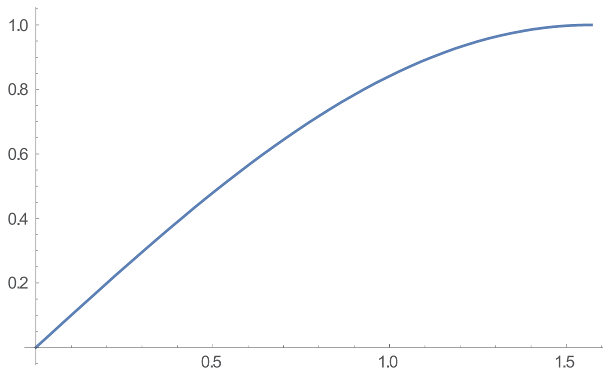
1. To reflect a graph through the y-axis means: replace each point (x,y) with the point (-x,y)
2. To reflect a graph through the x-axis means: replace each point (x,y) with the point (x,-y)
3. To reflect a graph through the origin means: replace each point (x,y) with the point (-x,-y)
4. To reflect a graph through the line y=x, replace each point (x,y) with the coordinates (y,x). (This is the inverse function.)

Exercise:

1. Below is the graph of a function, in the first quadrant. Find more parts of the graph if it is given to you that the graph is symmetric about the x-axis. Can you make a good estimate of what the function may be? Write the function if you can.



2. Below is the graph of a function for the range of x shown. Extend the graph if it is given to you that the function is symmetric about the line x=$\frac{\pi}{2}$. Can you guess what the function is and for what domain it is plotted? What are the values on the x-axis and y-axis?



Additional Fun Question: What is the inverse of the function y=2x? You can sketch and find out. Now also find out the inverse function of y=2x+1. Sketch it on the same graph.

Example 10

Topic: Forming and Solving Linear Equations

Level: Std X or higher

Lesson Objectives: Converting a word problem into a linear equation; Solving simultaneous linear equations; an introduction to matrix methods

Forming and Solving Linear Equations

You will often be given information in a written form or on a diagram and be required to find an unknown quantity. You need to form your own equation and then solve it to find the answer.

Exercises:

a) When x is doubled and 8 is added, the result is 26. Find the value of x. (Ans: 9)

b) Trebling (or tripling) a number and then taking away 5 gives the same result as doubling it and adding 2. What is the number? (Ans: 7)

c) If you subtract 6 from a number and then multiply the answer by 5, you get the same result as subtracting 4 and then multiplying by 3.

d) Ganesh, Bharat and Venkat have been collecting money for a charity fund. Ganesh has collected 3 times as much as Bharat and Venkat has collected Rs.420 more than Bharat. Altogether they have collected Rs.2870. How much money did each collect?

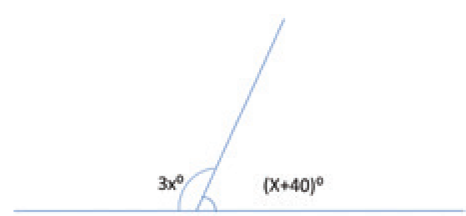
Hint: Let Bharat's money be x, then Ganesh has 3x and Venkat has x+60.

e) Mahesh is 3 years older than Sandra and Abdul is 2 years younger than Sandra. Together, their ages add up to 43 years. Find their individual ages.

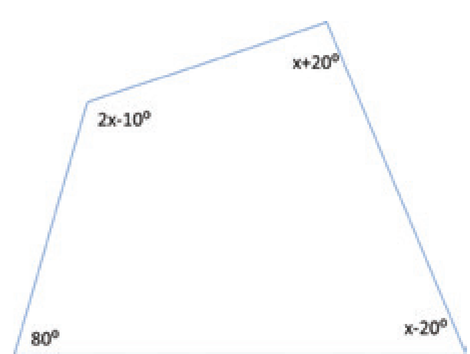
f) A bag contains white, grey and black counters. There are 14 more grey counters than white and 6 fewer black counters than white. There are 44 counters altogether.
   a. Write this as an algebraic equation
   b. Use this equation to find the number of white counters

g) Two angles of a triangle are 80° and x°. The third is $(30+x)^{\circ}$. Find x.

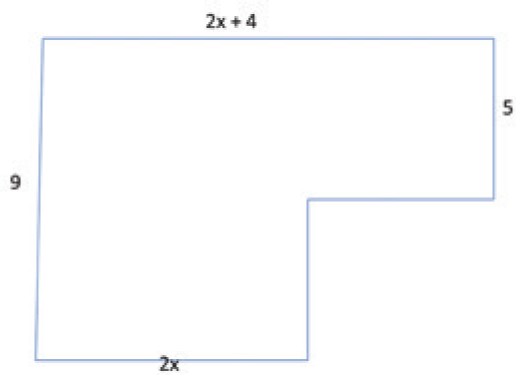
h)



i)



j) Find the value of x in this diagram if the perimeter of this shape is 38m.



k) Four identical triangles and a square are to be used to construct a pyramid. The base of each triangle is 5 cm wide and its height is (4x-3) cm. If the total surface area including the base is 115cm^2, then find the value of x.

SIMULTANEOUS LINEAR EQUATIONS

You know how to solve a linear equation involving one unknown. To solve it, you need to have one piece of information. E.g. a point on the line. That is, if the equation is $y = mx$, then using a point (x_1, y_1), the equation can be solved for x.

But if you have y=mx+c, you need two pieces of information to solve the line fully. That is, to find the values of the 2 unknown parameters, m and c, which are required to fully specify the line.

Similarly, if you have a linear equation in 3 variables (3 dimensions, for example the diagonal line of a cuboid), then you will have 3 unknown parameters, and 3 independent pieces of information about the cuboid will be required to solve it fully. A typical 3-d linear equation is of the form:

$$ax + by + cz = 0$$

You can solve 2 simultaneous equations by:

1. Elimination (Gaussian elimination)
2. Graphical method

You can only solve 3 simultaneous equations (in 3 dimensions, with variables x, y, z) by elimination method.

How many solutions?

The critical question in solving linear equations is: How many solutions will they have?

Case 1: Two linear equations in 2 variables. The general form of such linear equations is:

$$ax + by + c = 0$$

This has 2 parameters, a and b, which have to be found.

When there are 2 lines, several possibilities exist:
1. The lines intersect (in exactly one point)
2. The lines are parallel
3. The lines are overlapping (same line)

In each case, the number of solutions is different. This can be easily seen graphically. However, algebraic method will also give you clues as to the situation. See below table for the details:

Scenario	Number of distinct Solutions	Result of algebraic manipulation	Other clues
2 Intersecting Lines	1	Unique solution	There is no difficulty in solving the simultaneous equations
2 parallel lines	0	Ridiculous answer such as 2=0	Coefficient of x and y will be the same, only constant term will differ
Overlapping lines (same line)	Infinite	Useless answer such as 2=2	Coefficients of one line will be multiples of the coeffts of the other

Case 2: The same situation is true for linear equations in 3 dimensions (equation of a plane). Of course you require at least 3 independent equations to solve for 3 variables. If you have only 2 equations, you cannot solve for 3 variables.

See table below for the different situations with regard to 3 planes in 3-d space. Solving equations simultaneously means – looking for one or more points that satisfy all the equations. With 3 equations, we may have all the 3 planes intersecting at one point (like the corner of a room), or any pair of them intersecting at one point but the other one intersecting them at other points (like 3 sides of a cuboid, or triangular prism), or all planes non-intersecting (like the parallel sides of a cuboid), or all planes coinciding, that is, they are the same plane. The simultaneous equations give a unique result only in one case – that is, when they are concurrent. In all other cases, you will get some other result.

The typical equation will be:

$$ax + by + cz = 0$$

You should be prepared to get any of the above scenarios when you deal with simultaneous linear equations.

IMPORTANT NOTE:

The Triangular Form / Echelon Method/ Ladder Form/ Gaussian Elimination is the preferred algorithm for solving simultaneous linear equations. This is usually achieved by the use of Matrices. The matrix method is what is used in computers. The idea is to get the 2 or 3 simultaneous equations into a triangular form of the type:

$$px + qy + rz = s$$

$$0x + ty + az = b$$

$$0x + 0y + cz = d$$

Once the equations are in this form, it becomes very easy to solve.

Scenario	Number of solutions	Result of algebraic manipulation	Other clues
The planes intersect in one point (concurrent planes)	1	Unique solution is obtained by elimination process	No difficulty in solving; solution comes out consistently
The planes are not parallel, but do not have a common intersection point	0	Inconsistent equations. That means, the solution obtained by solving one pair of equations will not satisfy the third.	None
The planes are parallel	0	Ridiculous answer such as 2=0	Coefficient of x, y and z will be the same, only constant term will differ
The planes overlap or coincide (that is, they are the same plane)	Infinite	Useless answer such as 2=2	Coefficients of one equation will be multiples of the coeffts of the other

Example 11

Topic: Numbers in Different Bases

Level: Std XI or higher

Lesson Objectives: Place Values, Binary Notation and Problems

Base 10:

10^5	10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}
x100000	x10000	x1000	x100	x10	x1	x $\frac{1}{10}$	x $\frac{1}{100}$	x $\frac{1}{1000}$
5	2	6	7	9	5	3	1	8

This is the number 526795.318

Exercise:

Take a whole number for ease of understanding. Let the number (in base 10) be 3589.

$$N = N_{10} = 3589$$

$10^3 = 1000$	$10^2 = 100$	$10^1 = 10$	$10^0 = 1$
3x	5x	4x	9x

$$3 \times 10^3 = 3000\ (remainder\ 589)$$

$$+ 5 \times 10^2 = 500\ (total\ 3500, remainder\ 49)$$

$$+ 8 \times 10^1 = 80\ (total\ 3580, remainder\ 9)$$

$$+ 9 \times 10^0 = 9\ (total\ 3589, remainder\ 0)$$

This is how the number is structured.

We can use the same method to write the number in a different base, say base 9. Let the number be denoted by N_9

We shall use these results:

$$9^0 = 1$$

$$9^1 = 9$$

$$9^2 = 81$$

$$9^3 = 729$$

$$9^4 = 6561$$

$$0 \times 9^4\ (\because N < 9^4)$$

$$+ 4 \times 9^3 = 2916\ (total\ 2916, remainder\ 673 < 9^3)$$

$$+ 8 \times 9^2 = 648\ (remainder\ 25 < 9^2)$$

$$+ 2 \times 9^1 = 18\ (remainder\ 7 < 9^1)$$

$$+ 7 \times 9^0 = 7\ (remainder\ 0)$$

$$Total = 3589$$

Thus $N_9 = 4827$

What does this mean? In the base 9 notation, the decimal number 3859 would be written as 4827.

By the way, can you say how many digits would be used in this system? If you cannot, reattempt to answer this after going through the rest of this worksheet.

Now let us evaluate N_8

First, keep these handy:

$$8^0 = 1$$

$$8^1 = 8$$

$$8^2 = 64$$

$$8^3 = 512$$

$$8^4 = 4096$$

We will not need any more, since the next power of 8 will go well beyond our requirement.

$$0 \times 8^4 = 0\ (\because N < 8^4)$$

$$+ 5 \times 8^3 = 3584\ (total\ 3584, remainder = 5)$$

$$+ 5 \times 8^0 = 5\ (total\ 3589, remainder = 0)$$

That's it!

Thus $N_8 = 5005$

Interpret this for yourself. Say what it means.

$$N = N_{10} = 3589 = 3\times10^3 + 5\times10^2 + 8\times10^1 + 9\times10^0$$
$$= 5\times8^3 + 0\times8^2 + 0\times8^1 + 5\times8^0$$

That is what it means! It could have also been written as the sum of multiples of powers of 9, which we found earlier. Do that as an exercise.

Now let us evaluate N_7

Do you think N_7 will have more digits or the same as N_{10}?

Let us use:

$$7^0 = 1$$
$$7^1 = 7$$
$$7^2 = 49$$
$$7^3 = 343$$
$$7^4 = 2401$$
$$7^5 = 16807$$

$$1\times7^4 = 2401\ (rem = 1188)$$
$$+\ 3\times7^3 = 1029\ (rem = 159)$$
$$+3\times7^2 = 147(rem = 12)$$
$$+1\times7^1 = 7(rem = 5)$$
$$+5\times7^0 = 5(Total = 3589, rem = 0)$$

Thus $N_7 = 13315$

You may write this in polynomial notation in powers of 7 to understand it well!

What would be N_2 We start by making a table of powers of 2.

$$2^0 = 1$$
$$2^1 = 2$$
$$2^2 = 4$$
$$2^3 = 8$$
$$2^4 = 16$$
$$2^5 = 32$$
$$2^6 = 64$$
$$2^7 = 128$$
$$2^8 = 256$$
$$2^9 = 512$$
$$2^{10} = 1024$$
$$2^{11} = 2048$$

$$1\times2^{11} = 2048\ (rem = 1541)$$
$$+1\times2^{10} = 1024(rem = 517)$$
$$+1\times2^9 = 512(rem = 5)$$
$$+1\times2^2 = 4(\ rem = 1)$$
$$+1\times2^0 = 1(Total = 3589, rem = 0)$$

Thus $N_2 = 111000000101$

And that is how it is represented in a computer, which has a switch for each place, which is either on or off (denoting 0 and 1)!

This is called the binary representation, with only 2 symbols.

Try representing any other number to a base of your choice.

Can you use a base greater than 10? Can you work out 3589 written in the base 60?

(This may be taken as a simple research project. Some information can be found here.)

Further work:

Represent 6795.318 it in Base 9:

9^5	9^4	9^3	9^2	9^1	9^0	9^{-1}	9^{-2}	9^{-3}
X59049	X6561	X729	X81	X9	x1	x $\frac{1}{9}$	x $\frac{1}{81}$	x $\frac{1}{729}$
						.11111	.012345	.0013717
	1x	0x	2x	8x		2x	7x	7x
	6561		162	72		.22222	.08638	.0096
	Rem=234		Rem=72	0		Rem=.096	Rem=.009	Etc
0	1	0	2	8	0	2	7	7 etc.

The new number is 10280.277… (the fractional part may be evaluated further)

Example 12

Topic: Complex Numbers (advanced)

Level: Std XI or higher

Lesson Objectives: Introduction to Complex Numbers and their Geometrical Properties

COMPLEX NUMBERS – WHAT ARE THEY? WHAT DO THEY DO?

$$x^2 = -1$$

is the same as:

$$1 \times x^2 = -1$$

Which suggests that x is that transformation that, when repeated twice upon the number 1, takes it to -1.

From this we understand that the multiplicand

$$i = \sqrt{-1}$$

is the 90^0 rotation counterclockwise of a number on the real line.

If you take the unit circle, with the real number line along one diameter, the imaginary number line can be drawn along a transverse diameter. This gives the Argand Plane.

Exercise:
Take any point in the Argand Plane. Square it. What do you observe? Cube it. What do you observe?

Take another point. Do the same. Take the 5^{th} power.

Can you construct the point which makes 10^0 with the real axis, with unit magnitude? What will happen if it

Multiplication with a complex number is equivalent to (produces the same result as) rotation by a certain angle. Try out for different angles.

Problem:
Suppose a boat is traveling in a North-easterly direction such that it makes 4 miles to the North for every 3 miles to the East. Now it turns 45^0 anti-clockwise. What is its new orientation?

Solution:
In the argand plane, the multiplicand will move any point by rotation counterclockwise.

Our boat is at the point (3,4i) when it turns by 45^{0}counter-clockwise. A counter-clockwise rotation would correspond to multiplication with 1+ i.

The vector x+iy is a direction vector. Note, this gives orientation or direction, not location.

Worksheet 1

I. Algebra-based

1. Find the three cube-roots of unity by solving the equation $x^3 - 1 = 0$ algebraically.
2. Show that the three cube roots of unity sum to zero.
3. Show that one of the complex cube roots of unit is the square of the other, i.e. the roots are in the form: $1, \omega, \omega^2$
4. Solve the following equation: $x^2 + x + 1 = 0$
5. Form the equation whose roots are: $1 + i, 1 - i, 2$
6. If $1, \omega, \omega^2$ are the cube roots of unity, find the value of $\omega^7 + \omega^8 + \omega^9$
7. Without calculating evaluate $\frac{-b}{a}$ and $\frac{c}{a}$ if one root of the equation: $a\square^{\square} + bx + c = 0$ is $5i - 12$. Explain why the question cannot be answered if the given root is 2.
8. By solving the equation $x^3 + 1 = 0$, find the 3 cube roots of . If one of the complex cube roots is λ, express the other in terms of λ Prove that $1 + \lambda^2 = \lambda$.
9. Express as partial fractions with complex linear denominator: $\frac{2}{x^2+4}$
10. Find the complex factors of: $x^2 + 4x + 5$.

II. De Moivre's Theorem

11. Show that $Cos5\theta = Cos^5\theta - 10Cos^3\theta Sin^2\theta + 5Cos\theta Sin^4\theta$. Also find an identity for $Sin5\theta$ and for $tan5\theta$.
12. Prove that: $Sin^7\theta \equiv \frac{1}{64}\{35Sin\theta - 21Sin3\theta + 7Sin5\theta - Sin7\theta\}$. Hence find $\int[35Sin\theta - 64Sin^7\theta]\,d\theta$
13. Show that $tan4\theta \equiv \frac{4t-4t^3}{1-6t^2+t^4}$, where $t \equiv tan\theta$
14. Simplify: $\frac{\left(Cos\frac{\pi}{4}+iSin\frac{\pi}{4}\right)^5\left(Cos\frac{3\pi}{4}+iSin\frac{3\pi}{4}\right)^3}{\left(Cos\frac{\pi}{4}-iSin\frac{\pi}{4}\right)^2}$
15. Prove that: $tan6\theta \equiv 2t\left(\frac{3-10t^2+3t^4}{1-15t^2+15t^4-t^6}\right)$ where $t = tan\theta$
16. Prove: $Cos^4\theta + Sin^4\theta \equiv \frac{1}{4}(Cos4\theta + 3)$
17. Prove that: $tan3\theta = \frac{3tan\theta - tan^3\theta}{1-3tan^2\theta}$, and hence solve the equation: $1 - 3t^2 = 3t - t^3$. Given your answers to 3 significant figures and verify these results by an algebraic method.
18. Use de Moivre's theorem to find the following intergral: $\int 8Cos^4\theta d\theta$
19. As above: $\int(32Cos^6\theta - Cos6\theta)\,d\theta$
20. As above: $\int(Cos4\theta - 8Sin^4\theta)d\theta$

Worksheet #2

III. Complex Roots of Unity

1. If ω is a complex cube root of 1, simplify $(1 + \omega^1)(1 + \omega)$

2. If ω is a complex eighth root of unity, show that $\omega + \omega^7$ is real.

3. Find the 5th roots of unity. If ω is the root with the smallest positive argument and if $u = \omega + \omega^4$ and $v = \omega^2 + \omega^3$ then show that $u + v = 1$ and $u - v = \sqrt{5}$.

4. By considering the ninth roots of unity, show that $Cos\frac{2\pi}{9} + Cos\frac{4\pi}{9} + Cos\frac{6\pi}{9} + Cos\frac{8\pi}{9} = -\frac{1}{2}$

5. Prove that the 5th roots of 1 can be denoted by $1, \alpha, \alpha^2, \alpha^3, \alpha^4$ and show that $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$

6. If ω is a complex cube root of unity, simplify: $\begin{vmatrix} a & \omega b & \omega^2 c \\ \omega^2 b & c & \omega a \\ \omega c & \omega^2 a & b \end{vmatrix}$

7. If ω is a complex cube root of unity, simplify: $(1 + 6\omega)(1 + 6\omega^2)$

8. If β is a complex 4th root of unity, simplify: $(1 + \beta)(1 + \beta^2)$

9. Illustrate on an Argand diagram, the fifth roots of 1

10. If α is a complex 5th root of unity, simplify: $(1 + \alpha^4)(1 + \alpha^2)(1 + \alpha)$

nth roots of any number

Say we want to find 5th roots of 27.

$$27 = 27 \times 1 = 27 \times (\cos 2k\pi + i \sin 2k\pi)$$

The n complex roots of this number lie along a circle in the Argand plane. The radius of that circle has magnitude $27^{\frac{1}{5}}$, which is the real 5th root of 27. The exact complex root is determined by its position on the Argand plane, which will be along a circle of this magnitude.

Thus, the complex n^{th} roots of $r(Cos2k\pi + iSin2k\pi)$ are given by:

$$\left(r(Cos2k\pi + iSin2k\pi)\right)^{\frac{1}{n}} = r^{\frac{1}{n}} \times \left(Cos\frac{2k\pi}{n} + iSin\frac{2k\pi}{n}\right), k = 0,1,2,3 \ldots n-1$$

This means that if $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$ are the nth roots of unity, the nth roots of any other number R are : $r, r\omega, r\omega^2, r\omega^3, \ldots, r\omega^{n-1}$, where $r = R^{\frac{1}{n}}$, i.e. r is the real nth root of R. (E.g. 3 is the real 3rd root of 27).

That means, all nth roots can be found from the nth roots of unity! Likewise all nth roots of –ve numbers can be found from the nth roots of -1, by simply scaling the unit circle according to modulus = $R^{\frac{1}{n}}$! The nth roots of imaginary numbers (positive and negative) can likewise be found from the nth roots of $+i$ and $-i$ respectively, by just expanding or shrinking the unit circle!

Therefore, the complex roots of 27 are given by $27^{\frac{1}{5}} \times \left(Cos\frac{2k\pi}{n} + iSin\frac{2k\pi}{n}\right), k = 0,1,2,3,4$

Now, $27^{\frac{1}{5}} \approx 1.9332$, which is the real root, given by the value $k = 0$

The other roots are obtained from $k = 1$ and $k = 2$ along with their complex conjugates.

The roots are (check for yourself):
$1.9332, 0.60 + 1.84i, -1.57 + 1.12i, 0.60 - 1.84i, -1.57 - 1.12i$

Note that these are values that lie on the circle centre (0,0) and radius in the Argand Plane.

Pl plot these values on a graph paper and ascertain locations.

You may observe that this is larger than the unit circle.

Were we looking for the nth roots of 1/27, we would have found them on a circle smaller than the unit circle.

See visualization here: https://upload.wikimedia.org/wikipedia/commons/a/aa/Visualisation_complex_number_roots.svg

Example 13

Topic: Mathematical Induction

Level: Std XI or higher

Lesson Objectives: Problems and Solutions in Mathematical Induction

Question

Prove, by Mathematical Induction, that

$$(n+1)^2+(n+2)^2+(n+3)^2+\dots+(2n)^2=\frac{n(2n+1)(7n+1)}{6}$$

is true for all natural numbers n.

Discussion

Some readers may find it difficult to write the L.H.S. in P(k + 1). Some cannot factorize the L.H.S. and are forced to expand everything.

For P(1),

L.H.S. $= 2^2 = 4$, R.H.S. $= \frac{1\times 3\times 8}{6} = 4$. $\therefore$ P(1) is true.

Assume that P(k) is true for some natural number k, that is

$$(k+1)^2+(k+2)^2+(k+3)^2+\dots+(2k)^2=\frac{k(2k+1)(7k+1)}{6} \qquad \dots (1)$$

For P(k + 1) ,

$$(k+2)^2+(k+3)^2+\dots+(2k)^2+(2k+1)^2+(2k+2)^2$$

(There is a missing term in front and two more terms at the back.)

$$=(k+2)^2+(k+3)^2+\dots+(2k)^2+(2k+1)^2+4(k+1)^2$$

$$=(k+1)^2+(k+2)^2+(k+3)^2+\dots+(2k)^2+(2k+1)^2+3(k+1)^2$$

$$=\frac{k(2k+1)(7k+1)}{6}+(2k+1)^2+3(k+1)^2,\text{ by (1)}$$

$$=\frac{(2k+1)}{6}\left[k(7k+1)+6(2k+1)\right]+3(k+1)^2$$

(Combine the first two terms)

$$=\frac{(2k+1)}{6}\left[7k^2+13k+6\right]+3(k+1)^2$$

$$=\frac{(2k+1)}{6}(7k+6)(k+1)+3(k+1)^2$$

$$=\frac{(k+1)}{6}\left[(2k+1)(7k+6)+18(k+1)\right]$$

$$=\frac{(k+1)}{6}\left[14k^2+37k+24\right]$$

$$=\frac{(k+1)}{6}(2k+3)(7k+8)=\frac{(k+1)[2(k+1)+1][7(k+1)+1]}{6}$$

$\therefore$ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true for all natural numbers, n .

Question

Prove, by Mathematical Induction, that

$$1\cdot n+2(n-1)+3(n-2)+\dots+(n-2)\cdot 2+n\cdot 1=\frac{1}{6}n(n+1)(n+2)$$

is true for all natural numbers n.

Discussion

The "up and down" of the L.H.S. makes it difficult to find the middle term, but you can avoid this.

Solution

Let P(n) be the proposition:

$$1\cdot n+2(n-1)+3(n-2)+\dots+(n-2)\cdot 2+n\cdot 1=\frac{1}{6}n(n+1)(n+2)$$

For P(1),

L.H.S. = 1, R.H.S. $=\frac{1}{6}\times 1\times 2\times 3=1$. $\therefore$ P(1) is true.

Assume that P(k) is true for some natural number k, that is

$$1\cdot k+2(k-1)+3(k-2)+...+(k-2)\cdot 2+k\cdot 1=\frac{1}{6}k(k+1)(k+2) \quad .... (1)$$

For P(k + 1) ,

$$1\cdot(k+1)+2k+3(k-1)+...+(k-1)\cdot 3+k\cdot 2+(k+1)\cdot 1$$

$$=1\cdot(k+1)+2[(k-1)+1]+3[(k-2)+1]+ \dots+(k-1)\cdot[2+1]+k\cdot[1+1]+(k+1)\cdot 1$$

$$=1\cdot k+2(k-1)+3(k-2)+...+(k-2)\cdot 2+k\cdot 1$$
$$+1 \quad +2 \quad +3 \quad +...+ \quad (k-1) \quad +k \quad +(k+1)$$

(The bottom series is arithmetic)

$$=\frac{1}{6}k(k+1)(k+2)+\frac{1}{2}(k+1)(k+2) \text{ , by (1)}$$

$$=\frac{1}{6}k(k+1)(k+2)+\frac{1}{2}(k+1)(k+2) \text{ , by (1)}$$

$$=\frac{1}{6}(k+1)(k+2)[k+3]=\frac{1}{6}(k+1)[(k+1)+1]\,[(k+1)+2]$$

∴ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true for all natural numbers, n .

Question

Prove, by Mathematical Induction, that n(n + 1)(n + 2)(n + 3) is divisible by 24, for all natural numbers n.

Discussion

Mathematical Induction cannot be applied directly. Here we break the proposition into three parts. Also note that 24 = 4×3×2×1 = 4!

Solution

Let P(n) be the proposition:

1. n(n + 1) is divisible by 2! =2.
2. n(n + 1)(n + 2) is divisible by 3! = 6.
3. n(n + 1)(n + 2)(n + 3) is divisible by 4! = 24.

For P(1),

1. 1×2 = 2 is divisible by 2.
2. 1×2×3 = 6 is divisible by 3.
3. 1×2×3×4 = 24 is divisible by 24. ∴ P(1) is true.

Assume that P(k) is true for some natural number k, that is

1. k(k + 1) is divisible by 2, that is, k(k + 1) = 2a .... (1)
2. k(k + 1)(k + 2) is divisible by 6 , that is, k(k + 1)(k + 2) = 6b .... (2)
3. k(k + 1)(k + 2)(k + 3) is divisible by 24 , that is, k(k + 1)(k + 2)(k + 3) = 24c ....(3)

where a, b, c are natural numbers.

For P(k + 1) ,

1. (k + 1)(k + 2) = k(k + 1) + 2(k + 1)
   = 2a + 2(k + 1) , by (1) = 2 [a + k + 1] ....(4)

   which is divisible by 2.

2. (k + 1)(k + 2)(k + 3) = k(k + 1)(k + 2) + 3(k + 1)(k + 2)
   = 6b + 3 × 2[a + k + 1] , by (2), (4)
   = 6 [b + a + k + 1] .... (5)

   which is divisible by 6.

3. (k + 1)(k + 2)(k + 3)(k + 4) = k(k + 1)(k + 2)(k + 3) + 4(k + 1)(k + 2)(k + 3)

= 24c + 4 × 6[b + a + k + 1]   , by (3) , (5)
= 24 [c + b + a + k + 1]
, which is divisible by 24 .
∴ P(k + 1) is true.

By the Principle of Mathematical Induction, P(n) is true for all natural numbers, n .

Harder Problem :
Prove, by Mathematical Induction, that n(n + 1)(n + 2)(n + 3) …(n + r – 1) is divisible by r ! , for all natural numbers n, where r = 1, 2, ….

Prove that:

$$1.n + 2.(n-1) + 3.(n-2) + \cdots. + (n-1).2 + n.1 = \frac{1}{6}n(n+1)(n+2)$$

The non-MI method to prove a general problem of this kind is to find the general nth term.

Let's attempt it in this way first, just for fun. Examining the terms of this series, it is composed of two series:

$$Series\ I: 1, 2, 3 \ldots., n\ (kth\ term = k)$$

$$Series\ II: n, n-1, n-2, \ldots 1\ (kth\ term\ is\ n-k+1)$$

Thus the kth term of the combined series is:

$$T_k = k(n-k+1)$$

Then, the sum of the series is the summation of its kth term over the range of values of k, as follows:

$$S_n = \sum_1^n T_k = \sum_1^n k(n-k+1) = \sum_1^n (n+1)k - \sum_1^n k^2 = (n+1)\sum k - \sum k^2$$

Using the standard results on the sum of the first n natural numbers and the sum of the squares of the first n natural numbers, the RHS above is:

$$(n+1)\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} = \frac{n(n+1)}{6}[3(n+1)-(2n+1)] = \frac{1}{6}n(n+1)(n+2)$$

Q.E.D.

Now, let's see how to tackle this by Math Induction, using the method of segregating the terms. It's easier than we first thought!

Base case is established. By Induction Hypothesis, the foll statement is true:

$$P_k: 1.k + 2.(k-1) + 3(k-2) + \cdots + (k-1).2 + k.1 = \frac{1}{6}k(k+1)(k+2)$$

We wish to establish that the corresponding statement is true. LHS of this statement is:

$$1.(k+1) + 2.(k+1-1) + 3.(k+1-2) + \cdots + (k+1-2).3 + (k+1-1).2 + (k+1).1$$

The important thing is to note that this expression has (k+1) terms!

Write them out carefully.

Note that every term is a product: the first term goes up from 1 to k+1, the second term goes down from k+1 to 1. The number of terms is k+1. Now we can take out corresponding terms as Vikram has done:

LHS=

$$1.k + 2.(k-1) + 3(k-2) + \cdots + (k-1).2 + k.1$$
$$+ 1.1 + 2.1 + 3.1 + \cdots + (k-1).1 + (k).1 + (k+1).1$$

You should be able to see that we get the same pattern as with the other method, as we are splitting it into the LHS of P_k + Some additional terms.

Counting the terms correctly and maintaining the structure of *first_term x second_term* I think helps think this through properly. The 'fluctuation' occurs only when we fail to maintain the order: *first_term x second_term* consistently. So be careful at this stage, so that the problem does not become confusing.

Now it is clear that the additional terms are nothing but the sum of the first k+1 natural numbers, written in reverse order.

Thus, $LHS = \frac{1}{6}k(k+1)(k+2) + \frac{(k+1)(k+2)}{2}$, which upon simplification gives us the required result.

Thus LHS of $P_{k+1}=\frac{1}{6}(k+1)(k+2)(k+3)$, so that P_{k+1} is true. The rest of the reasoning of MI follows.

Example 14

Topic: Conic Sections – Implicit, Parametric and Polar Forms

Level: Std XI or higher

Lesson Objectives: Introduction to Complex Numbers and their Geometrical Properties

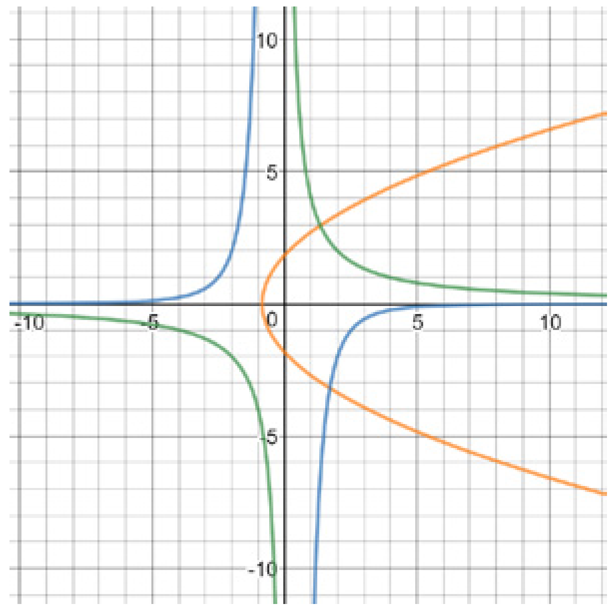
Implicit and Parametric Functions – First and Second-order Derivatives

Conic sections are a fertile ground for investigating the first and second order derivatives of functions which are presented in implicit or parametric forms.

The following are the standard forms of the conic sections in both implicit and parametric forms.

$\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ can be determined for all of these functions.

Plotting them on DesMos shows how complex their behaviour is. For example, alongside is the parabola $y^2 = 4x + 3.4$, along with the graph of its slope (the green hyperbola: $y' = \frac{4}{y}$) and the graph of the 'slope of slope' (the blue exponential curve: $y'' = \frac{-16}{y^3}$



Below are the general forms of the equations of the different conic sections. In each case $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ may be determined:

PARABOLA

Implicit form: $y^2 = 4ax + b$

(where the parameter *a* (focus) determines the location of the curve and *b* shifts it to left or right). You may check that:

$$\frac{dy}{dx} = \frac{a}{y}; \frac{d^2y}{dx^2} = \frac{-a^2}{y^3}$$

Parametric form: $x = at^2, y = 2at$

Here, $\dot{x} = 2at, \dot{y} = 2a; \frac{dy}{dx} = \frac{1}{t}; \frac{d^2y}{dx^2} = \frac{-1}{2at^3}$

ELLIPSE

Implicit form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Check that: $\frac{dy}{dx} = \frac{-b^2}{a^2}\frac{x}{y}; \frac{d^2y}{dx^2} = -\left(\frac{b}{a}\right)^2\left\{\left(\frac{bx}{ay}\right)^2 + 1\right\}$

Parametric form: $x = aSec\theta; y = b\, tan\theta$ Find $\frac{dy}{dx}; \frac{d^2y}{dx^2}$

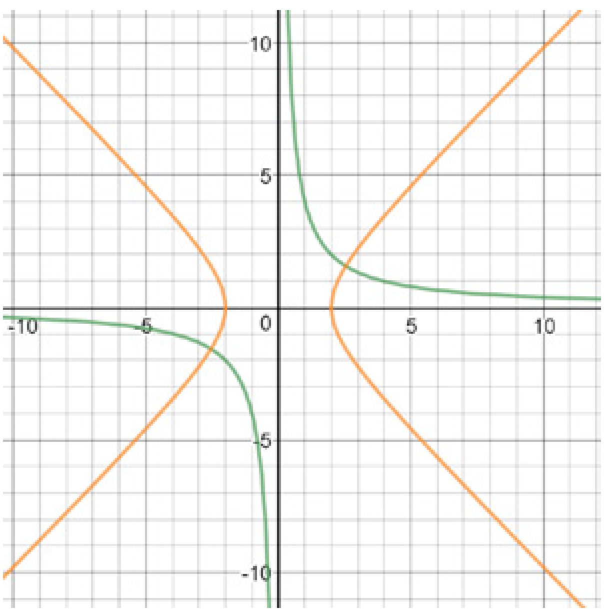
HYPERBOLA

Implicit form: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Check that: $\frac{dy}{dx} = \frac{b^2}{a^2}\frac{x}{y}; \frac{d^2y}{dx^2} = \left(\frac{b}{a}\right)^2\left\{\left(\frac{bx}{ay}\right)^2 + 1\right\}$

Parametric form: $x = aSec\theta; y = b\, tan\theta$ Find $\frac{dy}{dx}; \frac{d^2y}{dx^2}$

RECTANGULAR HYPERBOLA



Implicit form: $x^2 - y^2 = a^2\ OR, xy = k$

Below are the images for $x^2 - y^2 = 4\ (orange)$ $and\ xy = 4(green)$. The latter is just the rotated version of the former.

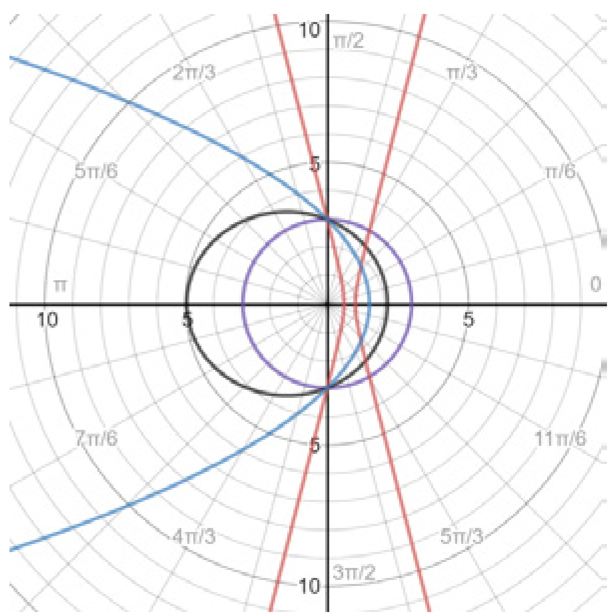
Parametric form: $x = ct; y = \frac{c}{t}$

Find $\frac{dy}{dx}; \frac{d^2y}{dx^2}$ in each case

Polar form of Conic Sections

It is useful to be aware of and try plotting the following conic sections (and lines) in the Polar Form.

Polar Form of Equation	Conic Section corresponding to equation
$r = a$	Circle
$r = \frac{a}{1 + Cos\theta}$	Parabola
$r = \frac{a}{1 + eCos\theta}, 0 < e < 1$	Ellipse
$r = \frac{a}{1 + ECos\theta}, E > 1$	Hyperbola
$rCos\theta = a$	Vertical line
$rSin\theta = a$	Horizontal line



Above are the graphs of

$r = 3(circle); r = \frac{3}{1+\cos\theta}(parabola); r = \frac{3}{1+.4Cos\theta}(ellipse)\ and\ r = \frac{3}{1+4Cos\theta}$

Notice how they all intersect at $\theta = \pm\frac{\pi}{2}$

Exercises:

1. Determine the area enclosed between $r = 2Sec\theta$ and $r = 1 + 2Cos\theta$.
2. Find the area enclosed between the parabola and the circle in the last diagram above.

Example 15

Topic: Sequences & Series

Level: Std XII

Lesson Objectives: Interesting convergent and divergent series and Problems

INTERESTING PROBLEMS ON CONVERGENCE TO PROVE

- The reciprocals of the positive integers produce a divergent series (harmonic series):
- Alternating the signs of the reciprocals of positive integers produces a convergent series:
- The reciprocals of prime numbers produce a divergent series (so the set of primes is “large”):
- The reciprocals of triangular numbers produce a convergent series:
- The reciprocals of factorials produce a convergent series (see e):
- The reciprocals of square numbers produce a convergent series (the Basel problem):
- The reciprocals of powers of 2 produce a convergent series (so the set of powers of 2 is “small”):
- Alternating the signs of reciprocals of powers of 2 also produces a convergent series:
- The reciprocals of Fibonacci numbers produce a convergent series (see ψ):

SUMMATION OF SERIES: WORKSHEET #1

1. Use of Standard Formulas
2. Method of differences

1. Obtain by any means, the sum of the first n natural numbers.

2. By summing both sides of the identity:
$$(r+1)^3 - r^3 = 3r^2 + 3r + 1$$
obtain an expression for the sum of the squares of the first n natural numbers.

3. Can you extend the above procedure to find the sum of the cubes, fourth powers, etc. of the first n natural numbers? What results do you need for each solution? What are your observations?

4. In problem #2, suppose we have r, r+1 and r+2 in place of r for the first term of LHS, and r-1, r, r+1 for the second term of LHS. Simplify. Thereafter, find an identity for the sum of the terms of the sequence $\{r(r+1)\}_{r=1}^{n}$.

5. Express $\frac{1}{(r-1)r}$ as partial fractions and hence find the sum of the series:
$$\frac{1}{(1)(2)} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \cdots + \frac{1}{(n-1)(n)}$$

6. If $f(r) \equiv r(r+1)!$, simplify $f(r) - f(r-1)$ and hence sum the series
$$5.2! + 10.3! + 17.4! + \cdots + (n^2+1).n!$$

7. Find $\sum_1^n r(r+1)(r+2$

8. Find the sum of the squares of the first n odd numbers

9. Find the sum of the cubes of the first n even numbers.

10. Given that $f(r) = \cos 2r\theta$, simplify $f(r+1) - f(r)$ and use your result to find the sum of the first 2n terms of the series $sin3\theta + sin5\theta + sin7\theta + \cdots$

SUMMATION OF SERIES WORKSHEET #2

Note that if $T_n = an^3 + bn^2 + cn + d$, then $S_n = a\sum n^3 + b\sum n^2 + c\sum n + nd$ and the summations indicated are all known via identities.

1. Find the nth term and sum to n terms of the series 3x5+4x7+5x9+…

   Ans: $\frac{4n^3+27n^2+59n}{6}$

2. Find the nth term and the sum to n terms of the following series:-

   a. $(1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \cdots$
   b. $1^2 + 2^2 + 3^2 + \cdots$
   c. $1.4.7 + 2.5.8 + 3.6.9 + \cdots$

3. Sum up the series: $1.n + 2.(n-1) + \cdots + n.1$

4. The natural numbers are grouped as follows:- Find an expression for the first term of the nth group.

   Ans: $T_n = \frac{n^2-n+2}{2}$

5. Find the nth term and deduce the sum to n terms using the method of differences: $4 + 11 + 22 + 37 + 56 + \cdots$

   Ans: $\frac{n}{6}(4n^2 + 9n + 1)$

6. As above, for the series: 1+5+13+29+…

   Ans: $2^{n+2} - 3n - 4$

7. Sum up to n terms: 5+55+555+…

8. Find the nth term, sum to n terms and sum to infinity of the series: $\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \cdots$

   Ans: $S_\infty = \frac{1}{6}$

9. Sum up to n terms: $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \cdots$

   Ans: $S_\infty = 2$

10. Sum up to n terms the series where the nth term is $2^n - 1$

    Ans: $2^{n+1} - 2 - n$

SUMMATION OF SERIES WORKSHEET #3

Convergence

Tests for Convergence

1. Determine whether the following series converge. If they do, give the sum to infinity.

   a. 3+5+7+…
   b. 1-1/4+1/16-1/64+…
   c. 3+ 9/2+ 27/4 +…

2. Find the condition on x so that $\sum_0^\infty \frac{(x-1)^r}{2^r}$ converges. Evaluate the expression when x=1.5.

3. Express the recurring decimal $0.1\dot{5}7\dot{6}$ as a fraction in its lowest terms. Hint: write as GP and find sum to inf.

   Ans: 35/222

4. The 3rd term of a convergent GP is the arithmetic mean of its 1st and 2nd terms. Find the common ratio, and if the first term is 1, find the sum to infinity.

5. Find the range of values for which the following series converge:

   a. $1 + x + x^2 + x^3 + \cdots$
   b. $x + 1 + \frac{1}{x} + \frac{1}{x^2} + \cdots$
   c. $1 + 2x + 3x^2 + 4x^3 + \cdots$
   d. $1 - (1 - x) + (1 - x)^2 - (1 - x)^3 + \cdots$
   e. $(a + x) + (a + x)^2 + (a + x)^3 + \cdots$
   f. $(a + x) - 1 + \frac{1}{(a+x)} - \frac{1}{(a+x)^2} + \ldots$

CONVERGENCE OF SERIES WORKSHEET #4

D'Alembert's Ratio Test

The methods for testing convergence that you are already familiar with are:

1. Direct summation of series, and testing the limit as $n \to \infty$

2. A) Comparison with known convergent/divergent series by taking the ratio $\frac{u_n}{a_n}$ and comparing the ratio with 1.

   B) Grouping of terms and then comparing.

There is a third method:

3. D'Alembert's Ratio test, where you take the limit of the ratio of (n+1)th term to nth term.

For a convergent series, D'Alembert's Ratio is less than 1 (strictly), for a divergent series it is strictly greater than 1. D'Alembert's test fails, however, when the ratio is exactly equal to 1.

In that case, you have to fall back on other methods, such as comparison test.

Note that the sequence in which you generally should apply the convergence tests is as follows:

First, try D'Alembert. If ration is > or < 1, you are done. Stop.

If it is =1, then try lim Sn. If you can find the sum to n terms, great, find the limit and you're done. Stop.

If you can't determine Sn, you may need to fall back on comparison test.

---

1. Prove that $\sum \frac{1}{n!}$ converges.

This problem can be done by comparison of the series:

$$S: 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$$

with the series:

$$R: 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \cdots$$

Taking the ratio of the nth terms:

$$\frac{u_n}{a_n} = \frac{1}{n!} \Big/ \frac{1}{2^{n-1}} = \frac{2^{n-1}}{n!} = \frac{1}{1} \times \frac{2}{2} \times \frac{2}{3} \times \ldots \times \frac{2}{n}$$

$$= \left\{\frac{1}{1} \times \frac{2}{2}\right\} \times \left\{\frac{2}{3} \times \ldots \times \frac{2}{n}\right\}$$

Here, the first bracket evaluates to 1; the second bracket is <1.

Therefore $\frac{u_n}{a_n} < 1$

Therefore, by comparison, the series S converges.

This can be done using D'Alembert's ratio as follows:

$$u_{n+1} = \frac{1}{(n+1)!}$$

$$u_n = \frac{1}{(n)!}$$

Then,

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1$$

Thus the ratio returns a value less than 1, indicating that the series converges.

You can now solve the more general result. Show that the following series is convergent:

$$\sum_n \frac{x^n}{n!}$$

Try this on your own by both methods above.

2. Show that the series $\sum_r \left(\frac{1}{r}\right)^k$ diverges when $k \le 1$.

   In this problem, note that the cases k=1, k=1/2 have both been already solved by you. k=1 gives the harmonic series.

   D'Alembert should be your first port of call and it will work.

3. Show that the series $\sum_r \left(\frac{1}{r}\right)^k$ converges when $k > 1$

   In this case, D'Alembert may not be able to help you (try it!)

   Grouping of terms may help you. As usual, leave the 1 as it is, group the next 2, then the next 4, etc.

   The comparator GP has $a = 1, r = \frac{1}{2^{q-1}}$

4. Using
   a) comparison test and
   b) differencing, prove that $\Sigma \frac{1}{r(r+1)}$ converges.
   Does the D'Alembert test help you here?

5. Test for convergence: $\Sigma \frac{n!}{(2n)!}$

6. Use a comparison test to show that if $\sum u_r$ is convergent, so is $\sum {u_r}^2$

7. Test for convergence: $\Sigma \frac{1}{n2^n}$

8. Test for convergence: $\Sigma \left(\frac{n+1}{n(n+2)}\right)^{\frac{1}{s2}}$

9. $\Sigma \frac{n}{2^n}$

10. $\Sigma \frac{n}{\sqrt{n^3+1}}$

7

Model Additional Scope

Key concepts and examples

- Methods of Proof
- Pigeonhole Principle
- Modular Arithmetic
- Graph Theory, Recurrence Relations (in brief)

Model exercises and references

1. Methods of Proof

Mathematics is said to be art of proof. The study of proofs and proof itself is the content of higher mathematics courses. There are many types of proofs. In particular, students must seek to be familiar with the following:

Distinction between axioms, rules of inference, theorem/non-theorem,

proofs...

Types of proofs:

- Direct, by exhaustion
- by Induction
- by Construction (vs. Existence only)
- by Contradiction
- by Contrapositive
- Computer assisted proof
- Probabilistic proofs
- Proof without words

Sample problem: Proof by Contradiction

Prove that $\sqrt{2}$ is an irrational number.

Solution: Suppose that $\sqrt{2} = \frac{p}{q}, q \neq 0, p, q \in \mathrm{Z}$

Further assume that the fraction $\frac{p}{q}$ is in its lowest terms. If it is not, it can be made to be. Therefore it is reasonable to assume that it is in its lowest terms.

Then $\left(\frac{p}{q}\right)^2 = 2, i.e. p^2 = 2q^2$

i.e p^2 is even, therefore p must be even too. (Why?)

Let $p = 2r$

Then $p^2 = (2r)^2 = 2q^2$

I.e. $4r^2 = 2q^2, or\ q^2 = 2r^2$

i.e. q is even too

If so, p and q have a common factor, 2.

This contradicts our supposition that p and q have no common factors.

Therefore. I.e. $\sqrt{2} \neq \frac{p}{q}, q \neq 0, p, q \in \mathrm{Z}$. I.e. $\sqrt{2}$ is an irrational number.

The above is an example of proof by contradiction.

Exercises:

1. Prove that there is no largest even number.
2. Show that the sum of a rational and an irrational number is irrational.

Additional Resources:
https://brilliant.org/wiki/contradiction/#writing-a-proof-by-contradiction

Proof by Contrapositive

Suppose that we are asked to prove a conditional statement, or a statement of the form "If A, then B." We know that we can try to prove it directly, which is always the more enlightening and preferred method. If a direct proof fails (or is too hard), we can try a contradiction proof, where we assume $\neg B$ and A, and we arrive at some sort of fallacy.

It's also possible to try a proof by contrapositive, which rests on the fact that a statement of the form "If A, then B." ($A \implies B$) is logically equivalent to "If $\neg B$, then $\neg A$." ($\neg B \implies \neg A$) The second statement is called the contrapositive of the first. Instead of proving that A implies B, you prove directly that $\neg B$ implies $\neg A$.

Example 1: Proof by Contrapositive

Let $x \in Z$. If $x^2 - 6x + 5$ is even, then prove that x is odd.

Proof. In this one, a direct proof would be awkward (and quite difficult).

Suppose that x is even. Then we want to show that $x^2 - 6x + 5$ is odd.

Write $x = 2a$ for some $a \in Z$, and plug in: $x^2 - 6x + 5 = (2a)^2 - 6(2a) + 5 = 4a^2 - 12a + 5 = 2(2a^2 - 6a + 2) + 1$.

Thus $x^2 - 6x + 5$ is odd. Proved.

Example 2: Prove by contrapositive:

Let $a, b, n \in Z$.If n does not divide ab, then n does not divide a and n does not divide b.

Proof: We need to find the contrapositive of the given statement.

First we need to negate "n does not divide a and n does not b." This is an example of a case where one has to be careful, as the negation is "n divides a or n divides b." (The "and" becomes an "or".)

The initial hypothesis is easy to negate: n divides ab.

Therefore, we are trying to prove "If $n \mid a$ or $n \mid b$, then $n \mid ab$." (Note: $n \mid a$ is read as 'n divides a') Suppose that n divides a. Then $a = nc$ for some $c \in Z$, and $ab = ncb = n(cb)$, so $n \mid ab$.

Similarly, if $n \mid b$, then $b = nd$ for some $d \in Z$, and $ab = and = n(ad)$, so $n \mid ab$.

Therefore, we have proven the result by contraposition.

Exercises:
Prove each of the following by the contrapositive method.

1. If x and y are two integers whose product is even, then at least one of the two must be even.
2. If x and y are two integers whose product is odd, then both must be odd.
3. If n is a positive integer such that n mod (3) = 2, then n is not a perfect square.

4. If a and b a real numbers such that the product a b is an irrational number, then either a or b must be an irrational number.

A Counting Proof:

Prove the uncountability of Real Numbers.

Students may wish to consult some of the resources mentioned in this document to find the proof. Instructors may be able to guide them on this problem.

Pigeonhole Principle:

If more than n objects are placed into n boxes, then at least one box must contain more than one object.

OR

If $n+1$ pigeons are placed in n pigeonholes, then there is a pigeonhole which contains at least two pigeons.

Understand the above principle. Try to re-phrase it. For example, can there be two pigeonholes with more than one pigeon in each? Can all the pigeons be placed in a single pigeonhole? Is the statement saying that there will be one pigeon in each pigeonhole or that there will be one pigeonhole with more than one pigeon?

This theorem is exemplified in real life by truisms like "in any group of three gloves there must be at least two left gloves or two right gloves".

Example 1:

How many students do you need in a school to guarantee that there are at least 2 students who have the same first two initials?

There are 26 X 26 = 676 different possible sets of two initials that can be obtained using the 26 letters A, B, C, ..., Z, so the number of students should be greater than 676. Thus, minimum number of students is 677.

Example 2:

You are given 11 different real numbers from 1 to 100 inclusive. Use the pigeonhole principle to show that two of those numbers differ by at most 10.

Note that the problem only asks you to show that there are two numbers in the 11 given numbers whose difference is not more than 10. It does not mean that the difference between any pair of numbers is not more than 10.

Solution. Divide the given interval into 10 equal parts.

They are: 1-10,11-20,21-30,…,91-100

Now place the given 11 numbers into their respective slots. By the Pigeonhole principle, since 11 numbers have to be placed in 10 slots, there will be a slot containing at least 2 numbers. Then these two numbers will differ by not more than 10.

Proved!

General form of Pigeonhole Principle:

If $kn+1$ objects are placed in groups, there is a group which contains at least k+1 objects

Proof by Contradiction:

Suppose that the statement is false. I.e. none of the n groups contains $k+1$ or more objects. That is, each of the n groups contains at the most k objects only.

Then the total number of objects can be at the most kn.

This contradicts the starting assumption about the number of objects. Hence we have a contradiction. Therefore the statement is not false, it is true.

Exercise:

3. A basket contains flowers of 5 different colours. How many flowers do you need to pick to be certain that you have 3 of the same colour?

4. Show that given a set of n positive integers, there exists a non-empty subset whose sum is divisible by n.

5. Nine points are marked on the inside of a square of side 2cm. Show that it is possible to find 3 points that form a triangle with area less than $1\ cm^2$.

Hint: First divide the square into 4 unit squares. Now, you have 9=8+1=2×4+1 points to place in each of 4 slots. Then there must be one which contains at least 2+1=3 points. And the triangle formed by any 3 of them must have area less than 1.

Study this topic further on: https://brilliant.org/wiki/pigeonhole-principle-definition/

2. MODULAR ARITHMETIC

If a number n gives remainder r when divided by then $n = kd + r$ for some number k. We say $n = r\ (mod\ d)$

An intuitive usage of modular arithmetic is with a 12-hour clock. If it is 10:00 now, then in 5 hours the clock will show 3:00 instead of 15:00. 3 is the remainder of 15 with a modulus of 12.

Example 1: If x gives remainder 6 and y gives remainder 3 when divided by 12, find the remainder when $x - y$ is divided by 12.

Solution:
Let $12p + 6, y = 12q + 3$, this is the given information

Then $x - y = 12(p - q) + 3 = 12r + 3\ say$.

This means $x - y = 3\ (mod\ 12)$. Solved.

Exercise:
1. If x gives remainder 5 when divided by 6, what remainder does 2x give?

2. If p gives remainder 4 when divided by 12 and q gives remainder 5 when divided by 12, what remainder does pq give?

Modular arithmetic has certain rules. We start with a definition:

We say a and b are *congruent modulo* m if m divides the difference $a - b$. Notation for congruence is $a \equiv b\ (mod\ m)$

2 and 4 are congruent modulo 2.

36 and 48 are congruent modulo 12.

If $a \equiv b\ (mod\ m)$, then both give the same remainder when divided by m. Note, quotient may be different.
Example: $\frac{36}{12} = 3, remainder\ 0\ and\ \frac{48}{12} = 4, remainder\ 0$

We can write $3 \equiv 5\ (mod 2)$ since both give remainder 1 when divided by 2

Further, if $a \equiv b\ (mod\ m)$ we can write $a = km + b$ or $a - b = km$. In the above example, 3=-1×2+5, and 3-5=-2=-1×2

Example: If $a \equiv b (mod\ m) and\ c \equiv d\ (mod\ m)$, $then\ a + c = (b + d)\ (mod\ m)$

Solution: Write

$$a = km + b$$
$$c = lm + d$$
$$a + c = (k + l)m + (b + d)$$

That is, $a + c = Km + (b + d)$

This is the same as $a + c \equiv (b + d)(mod\ m)$. Proved.

The other important arithmetic rules for modulo problems are:

If $a \equiv b\ (mod\ m), c \equiv d\ (mod\ m)$ then

1. $ka \equiv kb\ (mod\ m)$, for all integers k
2. $a + c \equiv (b + d)\ (mod\ m)$
3. $a - c \equiv b - d\ (mod\ m)$
4. $ac \equiv bd\ (mod\ m)$
5. $a^n \equiv b^n\ (mod\ m)$ for all naturals n

Example: Find the remainder when is 2^{30} divided by 13.

Solution: Factorize 30 as follows

$$2^{30} = (2^6)^5$$

Now $2^3 = 8$, and so $2^3 \equiv 8\ (mod\ 13)$

Therefore, applying rule number 5 from above:

$$2^{3^2} \equiv 8^2\ (mod\ 13) = 64\ (mod\ 13)$$

But $64\ (mod\ 13) = 12\ (mod\ 13) = -1\ (mod\ 13)$

Thus, $2^6 \equiv -1\ (mod\ 13)$

Therefore, $2^{6^5} \equiv -1^5 (mod\,13) = -1 (mod\,13) = 12 (mod\,13)$

Therefore the remainder is 12.

Try the same method for the following problems for practice:

Exercise:

Find the remainders:

1. $6^{15}\ (mod\ 12)$
2. $5^{22}\ (mod\ 25)$
3. $8 \times 3^{21}\ (mod\ 5)$

A Test For Perfect Squares

In this example, we will need two notions. An integer n is called a perfect square if there is another integer k such that n = k2. For example, 13689 is a perfect square since 13689 = 1172.
The second idea is the remainder and modular arithmetic.

For two integers m and n, n mod(m) = r will be the remainder resulting when we divide m into n. This means that there is an integer q such that n = mq + r.

For example, 127 mod(29) = 11 since 29 will go into 127 4 times with a remainder of 11 (or, in other words, 127 = (4)(29) + 11).

Determining whether or not a positve integer is a perfect square might be difficult. For example, is 82,642,834,671 a perfect square? First we compute 82,642,834,671 mod(4) = 3. Then use this theorem:

Theorem. If n is a positive integer such that n mod(4) is 2 or 3, then n is not a perfect square.

Proof. We will prove the contrapositive version: «If n is a perfect square then n mod(4) must be 0 or 1." (Do you understand why this is the contrapositive version?) Suppose $n = k^2$. There are four cases to consider.

1. If k mod(4) = 0, then k = 4q, for some integer q. Then, $n = k^2 = 16 q^2 = 4(4 q^2)$, i.e. n mod(4) = 0.

2. If k mod(4) = 1, then k = 4q + 1, for some integer q. Then, $n = k^2 = 16 q^2 + 8 q + 1= 4(4 q^2 + 2 q) + 1$, i.e. n mod(4) = 1.

3. If k mod(4) = 2, then k = 4q + 2, for some integer q. Then, $n = k^2 = 16 q^2 + 16 q + 4 = 4(4 q^2 + 4 q + 1)$, i.e. n mod(4) = 0.

4. If k mod(4) = 3, then k = 4q + 3, for some integer q. Then, $n = k^2 = 16 q^2 + 24 q + 9 = 4(4 q^2 + 6 q + 2) + 1$, i.e. n mod(4) = 1.

Additional Resources:
https://brilliant.org/wiki/modular-arithmetic/

For Graph Theory, the following introduction may be helpful:
https://medium.com/basecs/a-gentle-introduction-to-graph-theory-77969829ead8

From a computational standpoint, this may be useful:
https://www.tutorialspoint.com/parallel_algorithm/graph_algorithm.htm

For Recurrence Relations, again a brilliant resource:
https://brilliant.org/wiki/recurrence-relations/

8
Assessment Framework

Assessment framework for GLP-Math is best developed ground-up via learnings from the field. In principle, there can be no single criterion for testing and assessment. Gifted students learn at their own pace and of their own will, freely choosing the material to pursue. One clear measure of success has to be their performance in the school-based testing. Gifted students must needs be enabled to perform well in the school-based tests and examinations.

Learner assessment in the GLP is better assessed via indirect metrics such as 1. Topics studied; 2. Presentations made; 3. Papers written; 4. Project work; 5. Other external references to success/ achievement. Sustained activity in online thematic groups (for example in Math stack-exchange, or on brilliant.org, etc.) may be a far more effective measure of progress than a high score on a particular test. The online social / professional networks are self-selecting and self-appraising and survival in those groups is a robust measure of both aptitude and contribution. Being peer-evaluated, it is far more effective and puts the learner on a sustained growth path.

Successful blogs are another measure of growth and progress in the chosen profession/ line of study. Here is an example:

Madhav Sudarshan[7]

Please Visit My Blog: Flight and Beyond

And My Website: Aviation Unlimited, Inc.

Project work would be another very valuable source of insight into the student's progress. Selection of a project problem and progress in the direction of addressing it are very valuable indicators of achievement. Projects may be selected based on the student's aptitude and ability, as well as on the instructor's judgment about viability and availability of key resources. A good collection of student project ideas is available in the book titled Math Projects for Young Scientists (Thomas, 1988), although any number of project ideas can be found online. At Right Angles (Azim Premji University, 2012 onwards) is also an excellent resource for math projects.

The above examples show that assessment for the GLP-Math is complex and non-linear and may also vary from individual to individual as well as age-group to age-group.

Thus, assessment will have the following attributes:

1. Be primarily based on the learning objectives linked to core curriculum

2. Be achieved via topic-wise self-testing/ online tests and peer groups

3. Additionally incorporate:
   a. Project work
   b. Instructor's subjective appraisal of the

[7] MAIS student, batch of 2018

learner along 2 or 3 dimensions (e.g. effort, class participation, initiative)

4. Published work (targeting publications, conferences and events which accept student papers)

5. Competitions (such as Math Olympiads)

6. Post-school growth

As the assessment framework described above is dominated by lagging indicators, it is important for the instructor to stay in touch with the student (and parents), and to have an individualized progress chart for continuous monitoring. Students may leave the GLP at various points, when other things take primacy. It may not be appropriate to equate this kind of attrition with failure or poor performance. The success of the GLP may only be tracked over a long time-frame, e.g. including and beyond the student's college life and career. Significant choices and milestones in the student's life may be used as indicators of achievement, e.g. school performance, college admission, choice of profession, and professional success. Personal testimonies will also play a role in assessment of the learning programme.

Feedback from the instructors would be a critical factor in updating and rolling out the GLP-Math programme. Students may be found to benefit from particular types of interventions rather than other ones and it may also be found that different personality types play a role in these preferences. Thus, the focus may need to be more on 'opportunity provision' and 'guidance' than on goals, from the instructor's standpoint. Achievements may follow from the guided opportunities to learn, rather than from prescribed goals. The final test in this regard is the experience on the ground.

9

List of Resources

Institutional Resources:

At Right Angles, Azim Premji Foundation (indexed list of topics to be appended)

Over a dozen copies of At Right Angles, an excellent resource for students and teachers of mathematics, have been fully indexed with the following information.

This indexed list is available at Azim Premji Foundation.

A wealth of material is available in the articles of this journal, which can be purposed for classroom sessions by instructors. The material can also be independently studied by students. The article contributors constitute a fine set of resource persons for GLP-Math.

A small selecton from the indexed list is provided below. It may help the user to get an idea of the range and depth of the topics covered in the journal. Readers are encouraged to browse back issues of At Right Angles.

Topic	Sub-Topic	Grade	Keywords	Author	Issue
(Algebra/ Trig/ Calculus/ Geometry/ ..)	(Linear Equations/ Quadratics/ … Vectors/ Matrices…)	(Middle School/ High School/ …)	(Math History/ Proof/ Demonstration/ Tesellation/…)		Year Month (Page no.)

Name of Article	Issue	Syllabus Content	Class Addressed	Nature of Article	Author
Learning mathematics through puzzles	July 2014	Puzzle, game, solution, parity, invariance, proof, validity	Std 8 upwards	Games, analysis, communication, mathematization,	Rossi & Shikha
Bijaganita of Bhaskaracharya	Nov 2014	Fermat, Brahmagupta, Chakravala, Bhaskaracharya, infinity,number theory	Std 10 upwards	Awareness of : Indian mathematicians, mathematical verse, metre, anustup History of mathematics	Amartya Kumar Dutta
Paradoxes: True AND/OR False?	Mar 2015	truth value, Zeno, Ourobouros, ambigram, reflection, rotation	Std 10 upwards	self-reference, paradox, axiom, theorem, consistency, circular argument, proof, contradiction, illustration of math concepts	Punya Misra, Gaurav Bhatnagar

Name of Article	Issue	Syllabus Content	Class Addressed	Nature of Article	Author
Review of “ The Crest of the Peacock: Non-European Roots of Mathematics”	July 2015		Std 8 upwards	awareness, history of mathematics, Indian contribution, mathematical thinking, global nature, diverse transmissions	R Ramanujam
Magic Squares	Nov 2015	magic square, entry, prime, arithmetic progression	Std 9 upwards	creativity, reasoning, constraints	Vinay Nair
The Magical World of Infinities- Part 1	Mar 2016	Hilbert Hotel, Cantor, infinities, natural numbers, set of integers, set of rational numbers, one-one correspondence, interlacing, measure theory, cardinal numbers, cardinality	Std 10 upwards	Math vocabulary, history of mathematics, developments, analysis, symbolic representation	J. Shashidhar
The Difference-of-Two-Squares Formula: A New Look	July 2016	algebra, identities, alternative proofs, number line	Std 9 upwards		CoMaC Agnipratim Nag
Fractal Constructions Leading to Algebraic Thinking	Nov 2016	Algebra, extension, recursive, explicit, Sierpinski triangle, self-similarity, fractal construction, infinite, geometric progression	Std 9 upwards	investigation, technology in pedagogy, generalization, analogy, visualisation, linking across content areas	Jonaki Ghosh
Knot Theory	Mar 2017	Topology, knot, string, projection, tri-colourable, Reidemeister, unknot, trefoil, twist, poke, slide	Std 9 upwards	branches of mathematics, applications, student work	Ramya Ramalingam
Closing Bracket	Nov 2017		Std 9 upwards	How to nurture a love for math	Shailesh Shirali

Learning Websites:

A wide range of learning material is available on the internet. Students and instructors both need to be selective in choosing online material for study. Below is a list of excellent and reliable resources, used widely by students, math enthusiasts and teachers. Note that the material is not organized in any sort of prioritized manner. Users will be able to select their own favourite websites after browsing the below. Students are likely to already be familiar with many of the below and may have their own suggestions as to other good web-based resources. These lists may be reviewed and updated from time to time.

The below list is a compilation based on expert recommendations.

i. https://www.examsolutions.net/maths/

ii. http://www.khanacademy.org/

iii. https://brilliant.org/

iv. https://betterexplained.com/

v. https://twitter.com/preshtalwalkar?ref_src=twsrc%5Egoogle%7Ctwcamp%5Eserp%7Ctwgr%5Eauthor

vi. https://promys.org/resources/reading-list

vii. https://artofproblemsolving.com/

viii. https://www.awesomemath.org/

ix. http://hcssim.org/ (Hampshire summer school)

x. http://u.osu.edu/rossmath/ (Ross summer school)

xi. https://cty.jhu.edu/summer/index.html (John Hopkins' Gifted Students' program – Center for talented youth)

xii. http://math.mit.edu/research/highschool/primes/index.php (MIT's primes)

xiii. http://math.mit.edu/research/highschool/rsi/index.php (also MIT)

Technology:

xiv. https://www.geogebra.org/

xv. https://www.desmos.com/

xvi. https://www.wolfram.com/mathematica/

xvii. https://www.wolframalpha.com/

xviii. https://cocalc.com

Practice Tests, Challenge Tests and Question Banks:

xix. https://promys.org/program

xx. https://blogs.wsj.com/puzzle/category/varsity-math-2/

xxi. https://blogs.wsj.com/puzzle/2018/10/1/brain-games-6/

The above are two sets of puzzles published every week by Wall Street Journal in connection with Museum of Mathematics (New York)

xxii. https://momath.org/home/varsity-math/varsity-math-week-154/

xxiii. https://www.theguardian.com/science/series/alex-bellos-monday-puzzle

The above is Alex Bellos' weekly column in The Guardian

xxiv. https://www.expii.com/

The above is Poshen Lo's site

xxv. https://parallel.org.uk/

The above is Simon Singh's parallel site. A few problems are quoted elsewhere in this compilation.

Videos:

xxvi. http://www.numberphile.com/

xxvii. https://twitter.com/mathologer?lang=en

xxviii. https://dave.uktv.co.uk/shows/dara-o-briain-school-of-hard-sums/

xxix. https://www.youtube.com/user/MindYourDecisions

xxx. Movies about math: https://www.qedcat.com/moviemath/index.html#1

Craft:

xxxi. http://www.zometool.com/

The website discusses mathematical objects (and markets toolkits for making them).

xxxii. https://origami.ousaan.com/library/conste.html

There are many origami websites and online videos and books. This particular website discusses origami basics in the manner of the Axioms of Euclid.

10

Bibliography

Aggarwal, M. L. (2014). *Understanding ICSE Mathematics*. New Delhi: Avichal Publishing Company.

-- This is a standard textbook for ICSE curriculum for Classes IX and X.

Azim Premji University. (2012 onwards). At Right Angles. *At Right Angles*.

--At Right Angles is an excellent resource for students, teachers and math buffs of all age groups. A comprehensive listing of contents is provided in earlier pages.

Banerjee, A. D. (2010). *ICSE Mathematics for Class 10*. New Delhi: Bharati Bhawan.

Cambridge Assessment. (n.d.). *Cambridge Assessment International Education*. Retrieved January 2, 2018, from Cambridge Internationa: http://www.cambridgeinternational.org/

-- The detailed syllabus of the CIE is provided.

CBSE. (n.d.). *Central Board of Secondary Education*. Retrieved January 2, 2018, from CBSE: http://cbseacademic.nic.in/

-- The detailed syllabus of the CBSE is provided.

CISCE. (n.d.). *Council for Indian School Certificate Examinations*. Retrieved January 2, 2018, from www.cisce.org: http://www.cisce.org/isc_XII_Syllabus-2018.aspx

-- The detailed syllabus of the ICSE is provided.

Context Free Art. (n.d.). *Context Free Art*. Retrieved January 4, 2018, from Context Free Art: https://contextfreeart.org/index.html

-- An excellent resource for producing computer graphics of fractal forms. The code is simple and can be learnt by practice. The resulting images are not short of fantastic.

Gelca, T. A. (1956). *Mathematial Olympiad Challenges*. New Delhi: Springer (India) Pvt Ltd.

-- An excellent collection of math olympiad problems with solutions.

IBO. (n.d.). *International Baccalaureate*. Retrieved January 2, 2018, from IB Middle Years Programme: http://www.ibo.org/programmes/middle-years-programme/curriculum/

-- The detailed syllabus of the IB is provided.

Joan Franklin Smutny, E. (2003). *Designing and Developing Programs for Gifted Students*. California: Corwin Press.

-- A guide for educators designing and rolling out gifted education programmes, based on research done at Stanford University.

Mandelbrot, B. N. (1977). *The Fractal Geometry of Nature*. New York: W H Freeman & Company.

--The classic work of Benoit Mandelbrot is an enlightening read. It may not be possible to read it from cover to cover, but it is recommended to read the first few chapters.

National Board for Higher Mathematics. (n.d.). *NHBM*. Retrieved May 31, 2018, from Mathematics Olympiad: http://www.nbhm.dae.gov.in/olympiad.html

-- The manifesto of the NHBM with regard to the Indian math olympiad.

Patmore, R. P. (2000). *GCSE Mathematics.* London: Letts Educational.

-- A teaching book for Classes XI and X, based on the GCSE programme. Full of examples drawn from practical life.

Patwardhan. (2001). *Lilavati of Bhaskaracharya.* New Delhi: Motilal Banarsidass.

--Translation of the classic Sanskrit work Lilavati, dating back to a thousand years.

Paul Fannon, V. K. (2013). *Discrete Mathematics for the IB Diploma.* Cambridge: Cambridge University Press.

--An excellent classroom resource for topics in Discrete math such as Prime Numbers, Divisibility, Modulo, and Graph Theory.

Satyanarayana, B. (2015). *Some Problems from Algebra of Bhaskaracharya.*

Thomas, D. A. (1988). *Math Projects for Young Scientists.* New York: Franklin Watts.

TIFR. (n.d.). *Olympiads.* Retrieved May 31, 2018, from Question Papers and Solutions of INAO: http://olympiads.hbcse.tifr.res.in/how-to-prepare/past-papers/

Vishwambhar Pati, A. S. (2001). *Mathematical Analysis.* Hyderabad: Universities Press (India) Private Limited.

An excellent collection of papers from the journal Resonance, dealing with mathematics topics at the pre-University level, which will suitably challenge, inspire and push gifted learners.

11

APPENDIX

I. CALCULUS RESOURCES

One topic that is not covered extensively in this document by way of worksheets is Calculus. This is partly because standard school curricula extensively cover this topic. Students would benefit from material that show calculus in a 'new' light, for example, in the light of intuition. They would also benefit from exposure to visual representations of functions and from more exposure to applications. Resources for these purposes are listed below:

1. https://betterexplained.com/articles/prehistoric-calculus-discovering-pi/

2. https://www.khanacademy.org/math/calculus-home

II .SYLLABUS FOR MATHEMATICS CLASS IX TO CLASS XII

Topic	Sub-topics	Std IX	Std X	Std XI	Std XII	Advanced
Number Systems	Real Numbers	Natural numbers, integers, rational numbers	Real Numbers			
		Terminating/non-terminating recurring decimals	Euclid's division lemma			
		Nonrecurring/non terminating decimals such as 2, 3, 5 etc	Fundamental theorem of arithmetic			
		Existence of x for a given positive real number	Surds and irrationality			
		nth root of a real number	Decimal expansion of rationals			
Algebra	Commercial Math	Compound interest formula		Index numbers and moving averages	Matrices - concept and definitions	
		Interest compounded half-yearly		Complex Numbers	Operations on matrices	
		Rate of growth and depreciation		Conjugate, algebraic operations	Elementary row and column transformations	
	Exponents	Laws of exponents with integral powers		Argand diagram	Inverse	
		Rational exponents with positive real bases		Locus problems	Determinants	
	Rationalization	Rationalisation		Roots of unity	Minors, co-factors	
	Algebraic Identities	Expansions			Applications - area of triangle	
		Factorization			Adjoint and inverse of square matrix	
		Simultaneous Linear Equations			Solution of system of linear equations	Matrix as a transformation
	Polynomials	Polynomial in one variable	Zeroes of a polynomial		Consistent and inconsistent systems	Rank of a square matrix
		Constant, linear, quadratic, cubic polynomials; monomials, binomials, trinomials	Zeroes and coefficients of quadratic		Singular and non-singular matrices	Eigen values and eigen vectors
		Factors and multiples	Division algorithm for polynomial		Martin's rule for solving system of linear equations	Volume of parallelopiped
		Roots of a polynomial				Matrices and vectors
		Remainder Theorem				Geometrical interpretation of determinants
		Factor theorem				
		Algebraic expressions and identities				
	Binomial Theorem			Binomial theorem, Pascal's triangle		
				Proof using Math Induction		
	Permutations & Combinations			Fundamental principle of counting		
				Restricted permutations, word building, circular permutation		
				Combinations		
				Combinations - identities		
	Linear Eqn in 2 Var	Infinitely many solutions	Pairs of linear equations			
		plotting them and showing that they seem to lie on a line	Inconsistent equations			
		problems on Ratio and Proportion	Solution by substitution, elimination, cross multiplication			
		algebraic and graphical solutions				
	Quadratic Equations		Solution of quadratic equation by square completion	Qdtc equations, nature of roots		
				Qdtc functions, roots of given form		
				Sign of quadratic based on roots		
			Discriminant	Linear and qdtc inequalities		
			Nature of roots			
	Sequences and Series		Arithmetic progression	Arithmetic Progression - nth term, sum to n terms		
			nth term of AP, sum to n terms of AP	GP - nth term, sum to n terms		
				Geometric mean, arith mean		
				AGP - identifying series as AGP		
				Special sums, involving natural numbers		

Topic	Sub-topics	Std IX	Std X	Std XI	Std XII	Advanced
Coordinate Geometry	Coordinate Geometry	Point on Cartesian Plan	Lines in coordinate geometry	Basic concepts of points and coordinates		
		Notation	Distance formula	Straight line in detail		
				General equation of line		
				Locus		
		Linear equations and plotting	Area of triangle	Circles, different forms of equation		
		Introduce slope intercept form and general form		Tangents, condition for tangency		
				Equation of tangent, normal		
Geometry	Geometry	Axioms, Postulates, Theorems	Theorems and Proofs on similar triangles	Intro to 3-D geometry	3-D geometry	
	Circles		Theorems on tangents	As an extension of 2-D	Equations of lines and planes	
			Construction related to tangents and similar triangles	Distance formula	Direction cosines, direction ratios	
				Secton and mid-point form	Angle between lines	
					Cartesian and vector equation of line and plane	
					Skew lines, conditions for intersection	
					Distance of point from line	
					Shortest distance between skew lines	
					Equation to plane -- point form, normal form, intercept form	
					Distance of point from plane	
					Intersection of line and plane	
					Angle between planes, line and plane	
					Equation of plane through intersection of planes	

Topic	Sub-topics	Std IX	Std X	Std XI	Std XII	Advanced
Trigonometry	Trigonometry	Euclidean Theorems and Application	Trig ratios of acute angles of right triangle	Positive and negative angles		
		Mid-point theorem, equal-intercept theorem	Motivation	Radian measure and conversion		
		Pythagoras Theorem	Numerical values of standard trig ratios acute	Trig functions using unit circle		
		Rectilinear Figures	Basic trig identities	Trig identities		
		Triangles	Heights and distances (application of trig)	Signs of trig functions		
		Quadrilaterals		Domain and range of trig functions		
		Area theorems on Triangles and Parallelograms		Double angle and half-angle formulae		
		Circle definition and theorems		Angles and arc length		
		Chord properties		arc length formula		
		Arc and chord properties		Area of sector of circle		
		Constructions - bisectors, triangles		Trig functions of all angles, period of functions		
		Construction of Polygons		Graphs of trig functions		
				Compound and multiple angle formulas		
				Trig equations - general solution and solution in specified range		
				Properties of triangle - sine, cosine formula, area of triangle		
	Mensuration	Area of triangle - Herons formula		Conic sections as section of cone		
		Surface area and volume		Definition of focus, directrix, latus rectum		
				Parabola, equations, sketch		
				Find parabola when focus, directrix given		
				Ellipse, equation forms		
				Major axis, minor axis		
				Finding ellipse equation when focus, directrix given		
				Similarly for parabola		
				Condition that a line is a tangent to a conic		
				Locus problems		
Set Theory, Relations, Functions				Sets and representation		
				Venn diagrams, union, intersection, complement		
				Relations and functions		
				Domain, range,co-domain		
				Graph sketching		
				Types of functions (one-one, many-one, into, onto)		

Topic	Sub-topics	Std IX	Std X	Std XI	Std XII	Advanced
Calculus				Limits - notion and meaning	Continuity & Differentiability	Application of Calculus in commerce and economics
				Fundamental theorems	Derivative of composite functions	Cost function, demand func, revenue func, profit func
				Limits of algebraic and trig functions	Derivative of implicit and parametric functions	marginal cost and its interpretation
				Limits of exponential and log functions	Second order derivatives	marginal revenue and its interpretation
				Differentiation - meaning and geometrical interpretation	Rolle's Mean Value Theorem	Sketching curves and interpretation using maxima and minima
				Derivatives of simple algebraic and trig functions	Lagrange's mean value theorem (no proof)	Second order differential equations
				First principles	Lhospital's theorem	Reduction formulae
				Derivatives of sum, difference, product, quotient	Applications of derivatives	Surface area of solid of rotation
					Tangents and normals, maxima and minima	
					Integration	
					All standard methods of integration	
					Integration using partial fractions	
					Definite integral	
					Fundamental theorem of calculus	
					Differential equations	
					Area enclosed between curves	
					Differential equations, order and degree	
					Formation of differential equation	
					Solution of d.e.	
					Variable separable	
					Homogeneous equations	
					Linear form	
					Application - growth and decay	
					Population based problems	
					Applications in coordinate geometry	
Vectors					Vectors and scalars	Applications of scalar triple product
					Position vectors, components, I,j,k notation	Distance between skew lines
					Operations	
					Section formula	
					Triangle inequalities	
					Scalar product and geometrical significance	
					Vector product and its interpretation	
					Scalar triple product	

Topic	Sub-topics	Std IX	Std X	Std XI	Std XII	Advanced
Statistics & Probability	Statistics & Probability	Data - ungrouped, grouped		Mean, median, variance, standard deviation, IQR	Linear regression of x on y and y on x	Further probability distributions
		Tabulation		Mean deviation about mean and median	Scatter diagram	Linear combinations of Random Variables
		Bar graph, Histogram		Step deviation method	Least squares method	Hypothesis testing
		Frequency polygon		Mean, median and mode of grouped and ungrouped data	Best fit line	Non parametric tests
		Mean median Mode		Random experiments, outcomes, sample space, events	Identification of regression equations, terminology, notation of r-square	
		History of Probabilty		Exhaustive events, mutually exclusive events	Angle between regression lines	
		Frequency approach		Axiomatic probability theory	Estimation using regression lines	
		Activities		Definiton of probability	Conditional probability, bayes' theorem	
				Laws of probability, addition theorem	Independent and dependent events	
				Mean, quartile, decile, percentile and mode of gropuped and ungrouped data	Theorem of total probability	
				Combined mean and s.d.	random variables, expectation and variance	
				Correlation analysis	Bernoulli trials	
				Karl Pearson coefft of correlation	Binomial distribution	
				Rank correlation coefficient		
				Spearman's method		
Applied Math				Index numbers and moving averages	Introduction to LPP	
				Price index and price relative	Constraints, objective function, optimization	
				Aggregate methods	Uses or applications	
				Weighted average	Framing a LPP	
				Simple average of price	Graphical solution for 2 variable LPP	
				Moving averages	Feasible and infeasible solutions	
				Meaning and purpose of moving averages	Optimal feasible solution	
Mathematical Proof	Mathematical Proof	Axiom vs postulate	Mathematically acceptable statements	Principle of mathematical induction		
		Conjecture, theorem	Connecting words and phrases	Proof of summations, divisibility and inequalities		
		Deductive proof	Necessary and sufficient condition			
		Verification	Implies, exists, etc.			
		Counter-example	Difference between contradiction, converse and contra-positive			
Mathematical Modeling	Mathematical Modeling	Aims				
		Stages				
		Interpretation				
		Examples using Ratio, Proportion, Percentage				

ANITHA KURUP

Anitha Kurup is Dean and Professor, School of Social Sciences and Head of the Education Program at the National Institute of Advanced Studies (NIAS). She is currently leading the National Programme for the Gifted and Talented in India anchored at NIAS. As part of programme, she has developed Indian base protocols to identify and mentor the gifted children from diverse backgrounds-urban, rural and tribal areas. For the first time, in India, she has started weekend classes for the gifted children through the NIAS supported Advanced Learning Centres (NIAS-ALC). The other vertical that she is developing is creating advanced learning materials in multi-disciplinary areas of contemporary interests to support the gifted children. This resource book is the first one in this series. Her research interests span the broad disciplines of education and gender. She has made critical contributions to higher education and women in the STEM disciplines in India. Dr. Kurup has served as an expert in education in various capacities at the national and international levels. For details visit Weblink: http://nias.res.in/professor-and-dean/anitha-kurup

SHAILAJA D SHARMA

Shailaja D Sharma took her Ph.D. in 1990 from the Department of Mathematics, IIT Bombay. Subsequently, she worked for several years in the development sector as a Statistician, starting with the World Bank's District Primary Education Programme (DPEP). She then moved into the corporate sector and worked for over a decade with Royal Dutch Shell among others in senior business roles. A chance encounter with ancient Indian mathematical texts brought her irrevocably back into the world of mathematics and since 2015, she has immersed herself in mathematics and statistics education while also pursuing the study of the Indian mathematical tradition. Her teaching work has largely focused on students with high mathematical aptitude. She is a visiting faculty member at Azim Premji University and at Mallya Aditi International School, Adjunct Faculty at NIAS and a core-group member of the ICTS-NIAS Math Circles programme.

H S MANI

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