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Extension of Laguerre polynomials with negative arguments II

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Abstract

For integers $n, s, b_0, \dots, b_n$ with $n \geq 3, s \geq 0$, $|b_0| = |b_n| = 1$, let $G_1(x) = G_1(x, n, s) := n! \sum_{j=0}^{n} b_j (j!)^{-1} \binom{n+s-j}{n-j} x^j$. For $n \geq 0$ and $0 \leq s \leq 92$ it is proved in Shorey and Sinha (2022) that, except for finitely many pairs (n, s), $G_1(x) = G_1(x, n, s)$ is either irreducible or linear factor times an irreducible polynomial. If $s \leq 30$, we determine here explicitly the set of pairs (n, s) in the above assertion. This implies a new proof of the result of Nair and Shorey (2015) that $G_1(x)$ is irreducible for $s \leq 22$.

Keywords: Generalised Laguerre polynomials; Irreducibility; Primes; Valuations

1. Introduction

This is a continuation of [4]. Therefore we shall follow the notations of [4] but we shall recall here the key notations and key results from [4]. The generalised Laguerre polynomial of degree n with negative argument is

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{n} \frac{(\alpha + n) \dots (\alpha + j + 1)}{(n - j)!} \frac{(-x)^j}{j!}$$

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where α is negative. Then for $\alpha = -n - s - 1$ where s is a non-negative integer, we have

$$g(x) = g(x, n, s) := (-1)^n L_n^{(-n-s-1)}(x) = \sum_{j=0}^{n} a_j \frac{x^j}{j!}$$

where $a_j = \binom{n+s-j}{n-j}$ for $0 \le j \le n$.

Thus $a_n = 1$ and $a_0 = \binom{n+s}{n} = \frac{(n+1)\dots(n+s)}{s!}$ and

$$G(x) = G(x, n, s) := \sum_{j=0}^{n} \pi_j \frac{x^j}{j!} \qquad \text{where} \quad \pi_j = b_j a_j$$

such that $b_j \in \mathbb{Z}$ for $0 \le j \le n$ with $|b_0| = 1$, $|b_n| = 1$. For $k \ge 1$ we say we have (n, k, s) if $G(x) = G(x, n, s)$ has a factor of degree k and we do not have (n, k, s) if $G(x)$ has no factor of degree k. Next we write

$$g_1(x) = n!g(x), \quad G_1(x) = n!G(x).$$

Schur proved that $G_1(x)$ with $s = 0$ is irreducible. Therefore we always assume that $s > 0$.

2. Lemmas

In 1995, Filaseta [1, Lemma 2] gave the following lemma.

Lemma 1. *Let k and l be integers with $k > l \ge 0$. Suppose that $h(x) = \sum_{j=0}^{n} b_j x^j$ and p prime such that $p \nmid b_n$ and $p \mid b_j$ for $0 \le j < n - l$ and the right most edge of the Newton polygon for $h(x)$ with respect to p has slope less than $\frac{1}{k}$. Then for any $a_0, a_1, \dots, a_n \in \mathbb{Z}$ with $|a_0| = |a_n| = 1$, the polynomial $f(x) = \sum_{j=0}^{n} a_j b_j x^j \in \mathbb{Z}[x]$ cannot have a factor with degree in the interval $[l + 1, k]$.*

The next result is Lemma 1 from [4].

Lemma 2. *Assume that $G_1(x)$ has a factor of degree 1. Then*

$$n \le s^{\pi(s)}.$$

Further we state the following result from [4].

Lemma 3. *Let $n \ge 3$. Assume that $G_1(x)$ has a factor of degree $k \ge 2$. Then $s > 92$ unless*

$$\begin{aligned}(n, k, s) \in \{&(4, 2, 7), (4, 2, 23), (9, 2, 19), (9, 2, 47), (16, 2, 14),\\ &(16, 2, 34), (16, 2, 89), (9, 3, 47),\\ &(16, 3, 19), (10, 5, 4)\}.\end{aligned}$$

As an immediate consequence of Lemma 3, we derive the following result.

Lemma 4. *Let $n \ge 3$ and $s \le 92$. Except for finitely many triples*

$$\begin{aligned}(n, k, s) \in \{&(4, 2, 7), (4, 2, 23), (9, 2, 19), (9, 2, 47), (16, 2, 14),\\ &(16, 2, 34), (16, 2, 89), (9, 3, 47),\\ &(16, 3, 19), (10, 5, 4)\},\end{aligned}$$

$G_1(x)$ is either irreducible or

$$G_1(x) = (x - \alpha)H_1(x) \tag{1}$$

for some uniquely determined $\alpha \in \mathbb{Z}$ and monic irreducible polynomial $H_1(x) \in \mathbb{Z}[x]$.

Proof. Let $s \leq 92$. Assume that $G_1(x)$ is reducible. Then we derive from Lemma 3 that either (n, k, s) belongs to the finite set stated in Lemma 3 or $G_1(x)$ has no factor of degree $k \geq 2$. Now the assertion follows immediately. □

3. Irreducibility of $G_1(x, 2, s)$ for $s \in \{3, 7, 15\}$

We compute

$$G_1(x) = b_2x^2 - 2(1+s)b_1x + b_0\frac{(2+s)(1+s)}{2} \tag{2}$$

where $|b_0| = |b_2| = 1$. For the irreducibility of $G_1(x)$ it suffices to show that the polynomials

$$x^2 \pm 2(1+s)b_1x \pm \frac{(2+s)(1+s)}{2}$$

are irreducible. We prove

Lemma 5. *The polynomials (2) with $s = 3$ and $s = 15$ are irreducible for every $b_1 \in \mathbb{Z}$. Also the polynomial (2) with $s = 7$ is irreducible for every $b_1 \in \mathbb{Z}$ except for $b_1 = 0$ where the polynomial is $x^2 - 36$.*

Proof. The proof depends on a well known assertion that a quadratic polynomial is irreducible if and only if its discriminant is not a square. We consider $x^2 - 8b_1x + 10$ obtained from (2) by putting $b_0 = 1 = b_2$. Suppose it is reducible. Then its discriminant $(8b_1)^2 - 40 = m^2$ for an integer $m \geq 0$. Thus $(8b_1 - m, 8b_1 + m) \in \{(1, 40), (2, 20), (4, 10), (5, 8)\}$ and then $16b_1 \in \{41, 22, 14, 13\}$. This is not possible since none of $41, 22, 14, 13$ is divisible by 16. The assertion follows similarly for all other cases. □

4. $G_1(x)$ divisible by a linear factor

For $s \leq 92$, we see from Lemma 4 that except for finitely many cases, $G_1(x)$ is either irreducible or divisible by a linear factor. In this section, we consider the case where $G_1(x)$ is divisible by a linear factor. Then we derive from Lemma 3 that n is bounded by a computable number depending only on s. If s is restricted to 30, we prove a more precise assertion.

Theorem 1. *Let $n \geq 2, s \leq 30$ and $G_1(x, 2, 7) \neq x^2 - 36$. Assume that $G_1(x) = G_1(x, n, s)$ is divisible by a linear factor and*

$$(n, k, s) \notin \{(4, 2, 7), (4, 2, 23), (9, 2, 19), (16, 2, 14), (16, 3, 19), (10, 5, 4)\}. \tag{3}$$

Then $(n, s) \in X$ where

$$X = \{(6, 3), (4, 5), (8, 11), (72, 11), (3, 15), (10, 15), (4, 15), (12, 15), (8, 15), (16, 17), (272, 17), (8, 27), (16, 29), (786600, 25), (786600, 26)\}.$$

Proof. By definition, the assumption (3) is interpreted as $G_1(x)$ has no factor of degree 2 at $(n, s) \in \{(4, 7), (4, 23), (9, 19), (16, 14)\}$, no factor of degree 3 at $(n, s) = (16, 19)$ and no factor of degree 5 at $(n, s) = (10, 4)$. Assume that $G_1(x)$ is divisible by a linear factor. Then, as in [4, Lemma 2], we have

$$n = \prod_{p|n} p^{\nu_p(n)} = \prod_{p\le s} p^{\nu_p(n)} \tag{4}$$

where

$$p^{\nu_p(n)} \le s \quad \text{for} \quad p \le s \tag{5}$$

and

$$p \mid \frac{(n+1)\dots(n+s)}{s!} \quad \text{for} \quad p \mid n. \tag{6}$$

(5) follows from (6) and [4, Lemma 2]. Denote by T the set of all pairs (n, s) satisfying (4), (5) and (6). By applying Lemma 1 with $l = 0, k = 1$ to all pairs $(n, s) \in T$, we check that Lemma 1 does not hold for the following set T_1 of pairs (n, s) given by

$$\begin{aligned}
&\{(2, 3), (6, 3), (4, 5), (2, 7), (4, 7), (8, 11), (72, 11), (8, 13),\\
&\quad (3, 15), (2, 15), (10, 15), (4, 15), (12, 15),\\
&(8, 15), (16, 17), (272, 17), (16, 19), (6, 23), (4, 23), (16, 23),\\
&\quad (16, 24), (16, 26), (8, 27), (216, 29),\\
&(16, 19), (786600, 25), (786600, 26)\}.
\end{aligned}$$

Denote by T_2 the pairs (n, s) with $n = 2$. These are excluded by Lemma 5. Denote by T_3 the complement of $T_2 \cup \{(3, 15)\}$ in T_1. Then all the pairs $(n, s) \in T_1$ satisfy $n \ge 4$. Therefore we derive (1) uniquely for every $(n, s) \in T_3$ by Lemma 4. Denote by T_4 the set obtained by applying Lemma 1 with $l = 1$ and $k = [\frac{n}{2}]$ to $G_1(x)$ with $(n, s) \in T_3$. We calculate $T_4 = X \setminus \{(3, 15)\}$. Now the assertion of Theorem 1 follows immediately. □

Now we give an application of Theorem 1 with $G_1(x)$ replaced by $g_1(x)$. We prove

Corollary 1. *Let $s \le 30$. If $g_1(x)$ is reducible, the*

$$(n, s) \in \{(786600, 25), (786600, 26)\}.$$

This implies that $g_1(x)$ with $s \le 24$ is irreducible which includes a new proof of a result of Nair and Shorey [3]. We refer to [4] for a complete account of results proved on the irreducibility of $g_1(x)$. The results of Hajir and its refinement by Nair and Shorey and Jindal, Laishram and Sarma depend on algebraic results of Hajir [2] on the Newton polygons. Our proof of Corollary 1 is new in the sense that it does not use the above results of Hajir [2] on Newton polygons.

Proof of Corollary 1. Let $s \le 30$ and $G_1(x) = g_1(x)$ be reducible. We compute that $g_1(x)$ is irreducible for $(n, s) \in \{(4, 7), (4, 23), (9, 19), (16, 14), (16, 19), (10, 4)\}$. Now we derive from Lemma 3 that $g_1(x)$ is divisible by a linear factor. We verify that $g_1(x)$ is irreducible for $(n, s) = (2, 7)$. Therefore, the assumptions of Theorem 1 with G_1 replaced by g_1 are satisfied. Hence we conclude $(n, s) \in X$ by Theorem 1. Now we compute $g_1(x)$ with $(n, s) \in X$ are irreducible. This is a contradiction since $g_1(x)$ is divisible by a linear factor. □

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References

[1] M. Filaseta, The irreducibility of all but finitely many bessel polynomials, Acta Math. 174 (1995) 383–397.
[2] F. Hajir, Algebraic properties of a family of generalized laguerre polynomials, Canad. J. Math. 61 (2009) 583–603.
[3] G. Nair Saranya, T.N. Shorey, Irreducibility of Laguerre polynomial $L_n^{(-n-s-1)}(x)$, Indag. Math. (N.S.) 26 (2015) 615–625.
[4] T.N. Shorey, Sinha Sneh Bala, Extension of Laguerre polynomials with negative arguments, Indag. Math. (N.S.) (2022) https://doi.org/10.1016/j.indag.2022.02.006, (in press).