Duality in science

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Scientific concepts are unique in the way they accrue multiple meanings. This feature will be illustrated through the powerful concept of duality in science. We discuss the multiple meanings of this concept in different domains of science and analyse the duality relations found in them. The relationship between duality and symmetry, as well as with the notion of completeness, is explored to understand the many uses of the concept of duality.

CREATION of concepts is not unique to the activity of science, but the types of concepts that are created are indeed unique and different from the ones in other disciplines. A scientific description is not only related to methods such as observation and experiment, but it also crucially depends on the concepts it uses to describe phenomena. One could even mention that without the scope of these concepts it would be quite impossible to make a phenomenon 'scientific'. The activity of science such as theorizing, and experimenting, etc. involves not only creating new concepts but also tinkering with the available ones. Even though the need of novel concepts for specific formulations was known right from the early stages of science, only in the later half of the 20th century, largely through the works of Thomas Kuhn, it was recognized that scientific development necessarily entails modification of concepts. To mention a well-known instance of this, consider the shift in conceptualizing mass from Galileo to Newton. Galileo was the first to propose the definition of science as an activity which involves describing phenomena using measurable concepts. In order to demonstrate this approach, he borrowed concepts like mass, space, time, etc. which were already available within the mechanical philosophy during his time, and modified them suitably. This choice of what type of concept is needed to do science influenced the growth of science post-Galileo. Newton added another level of complexity to this strategy by suggesting that scientific phenomena and objects are essentially reducible to measurable concepts. This approach to the description of scientific objects has become a powerful tool in the practice of science. For example, when we consider an electron, all that is needed for a scientific description of the same is to give its essential attributes like its mass,

charge and spin. Other descriptions such as 'what is it really made up of', and so on, are considered irrelevant.

Measurable properties are just one kind of scientific concept. There are other concepts that cannot be characterized through quantitative values. Some of these capture structural reality of scientific descriptions. A well-known representative of this type of concept is symmetry which is found in all the branches of science. Symmetry, as a broad concept, is not reducible to 'measurable' values, but at the same time the definition of symmetry as invariance under transformations allows individuated descriptions of it. Symmetries can be classified by the specific groups under which they remain invariant and thus the general concept of symmetry is a good example of an extended meaning of 'measurability' that is necessarily associated with scientific concepts.

Here we focus on another intriguing scientific concept: duality. This concept is encountered in various scientific disciplines and surprisingly, in many of these contexts, duality seems to have different meanings. This concept also plays a significant role in non-scientific disciplines like philosophy, social science and anthropology. In philosophy, among many different uses, an important one is dualism, which signifies an ontological position that is opposed to monism and pluralism. Well-known instances of this stance are Descartes' duality of mind and body, and the duality of brahman and atman in Vedanta philosophy. In anthropology, Claude Levi-Strauss[1], based on his fieldwork in South America, found support for the view that conceptualization of binaries characterized human thought in all societies, including isolated tribal groups. Binary, which is one of the connotations of duality, typically denotes two concepts that are 'opposite' to each other, e.g. 'light and dark', 'man and woman', 'good and evil', etc. It could be argued that this same universal characteristic of human thought, namely the tendency to describe the conceptual world in terms of binaries, is also carried over into the scientific domain. This should not be surprising since we do expect that forms of thought are not radically different in science and other domains. For

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instance, Gerald Holton[2], while discussing ways of con-
textualizing and analysing scientific works, notes how
these can be uniquely understood through thematic
conceptual pairs – like reductionism–holism, hierarchy–
unity, concrete–abstract, evolution–devolution, discrete–
continuum – that constrain and motivate these enquiries.

We will first begin by summarizing some of the usages
of duality in science. Subsequently, we will analyse these
varied uses and isolate a few important characteristics of
these dualities. Through this analysis, we also want to
make evident an important feature of scientific thinking
found in these disciplines: the tendency to take an already
available concept and use it in a specific context. It is this
usage of a concept in varied contexts that results in mul-
tiple connotations getting associated with it.

Different invocations of duality

What is duality? Given that 'duality' is also used fre-
quently in everyday, ordinary contexts, is there a general
meaning of this concept that we are already acquainted
with? It is useful to consider this general notion before
more technical ones are discussed. The basic notion of
duality usually involves just two things conjoined by
connectives like 'and', 'or', or even a hyphen. This
description of duality is not specific about the relation
that binds two things and hence works with the minimum
requirement of having two things. This basic notion
captures an essential aspect found in all notions of duality
and that has to do with the cardinality – there being just
the right number of things, namely two and not one or
three. In this basic form, therefore, what matters to duali-
ty is nothing more than the count of the things involved
irrespective of how these two things are related, which
might be through conjunction, disjunction, or any other
form. Even though this notion of duality is trivial since
any two things on an ad hoc basis can be brought together,
we commonly use a form of cardinal duality while cata-
loguing and classifying the world around us. This slightly
restrictive form refers to the duality usually observed in
specific kinds of things, e.g. 'pair of scissors' and 'biped
animals'. Usage of duality in this context attempts to
capture those things that usually come in pairs. A well-
known example is the presumption of dual biological
sexes: male and female. The underlying presumption of
'sexual duality' shaped the development of biology in
the last two centuries[3]. Another historically interesting,
although not so well-known instance of this usage of dua-
lity is found in a proposal about human brains in the 19th
century. The anatomical discovery of human brain having
two hemispheres gave raise to the speculation that we do
not have a single, unified brain. Instead, we possess 'dual
brains' and 'dual minds'[4]. According to this proposal, it
is 'entirely unphilosophical ... to speak of the cerebrum as
one organ. The term two hemispheres of the brain is
indeed, strictly a misnomer ... The two hemispheres are

really and in fact two distinct and entire organs, and each
respectively as complete ... and as fully perfect in all its
parts ... as are the two eyes'[4].

In contrast to the above mentioned general usages,
invocations of duality found in logic, mathematics and
other disciplines of science are more complex and inter-
esting. These specific instances of duality, however,
appear schematically in forms that do not much differ
from the ordinary instances. That is, in the articulation of
dualities found in category theory or one between two
theories of physics, the relation that stands between duals
seems to be similar to the simple connections that were
discussed above. In spite of this superficial similarity, the
instances of duality found in science are much
more complex. As we will show below, the entire task of
understanding each of these dualities involves interpret-
ing the seemingly straightforward connectors ('and', 'or',
and '–') that exist between the duals. In fact, it is not
difficult to show how the relation between two things is
something more than the connection that is represented
between them. To see this we only need to note the use of
common phrases like 'salt and pepper', 'good or bad', 'by
and large', 'mother and kid', and many more used in
English (and similar kinds of phrases in other languages).
These phrases are not too unfamiliar as they are encoun-
tered regularly in communication. However, these are not
instances of simple duals as the relation between the two
words involved is something more than the obvious
connection that is present between them. Labelled as
binomials in linguistics, the phrases consists of two words
that are related not only by some syntactic relationship,
but they also share several semantic and phonological
relationships between them[5]. These relations that bino-
mials share, more than the obvious 'and' or 'or' that
exists between them, in fact shape important characteris-
tics like the order in which the binomials appear and also
the flexibility of their order.

The above segue into linguistics hopefully presents the
need for going beyond the obvious surface-level relation
that appears between the duals. Therefore, when the
commonly used phrase 'wave–particle duality' is invoked,
it is obviously not clear how 'wave' and 'particle' are
related by the connector. More analysis is needed to
know whether it is mere conjunction that constitutes the
duality, or whether there is something beyond this. There-
fore, in order to understand the notion of duality and to
clearly define it, we will first discuss several kinds of
duality that are present in logic, mathematics and other
areas. All these instances of duality are much more than
mere juxtaposition of two entities.

Duality in logic

In the case of classical logic, duality is contextualized
amidst truth functions. A truth function is a statement
whose truth value (i.e. whether it is true or false) depends

on the truth values of constituent statements and how they are connected through logical connectives like AND, OR, NOT, etc. The best way to introduce this kind of duality is by illustrating how simple conjunction and disjunction truth functions (i.e. statements constituted by the AND and OR operators) constitute an instance of logical duality. These statements are represented schematically as '$p \wedge q$' and '$p \vee q$' respectively, where p and q are simple statements. These schemata are considered duals of each other because the truth tables of these functions show a unique similarity. The similarity between the truth tables of these two statements is not directly evident, as can be seen in Tables 1 and 2. However, under specific transformation, these truth tables show similarities. To demonstrate the same, consider, the table of conjunction. In this, if the truth values are swapped in all the columns, we get a table that has the truth-value distribution as shown in Table 3. This resultant table, when a few rows are rearranged, is nothing but the truth table of the disjunction operation. Therefore, even though the disjunction and conjunction schemata have different truth tables to begin with, these tables turn out to be similar under a specific transformation. This indicates some structural similarity between these two schemata. So, it is the simultaneous consideration that these two truth functions are unique and yet are similar to one another after some alteration which is labelled as logical duality. According to Quine, duals are 'alike under truth-value analysis except for a thoroughgoing interchange' of T and F in truth tables[6].

Duality in mathematics

There are numerous instances of duality in mathematics. In fact, one of the earliest articulations of 'duality' comes from projective geometry. The pervasiveness of duality in mathematics has been observed and discussed by several mathematicians[7]. Also, several philosophers of mathematics have attempted to provide unified theories of these duality instances found in different branches of mathematics[8]. In this section, we will highlight a few important instances of duality.

Projective geometry: In projective geometry, the sets of axioms, the implied theorems and the configurations (i.e. the geometric figures) are articulated largely through the primitive concepts like lines, points, incidence and separation[9]. Given this, a particular kind of duality is observed that arises due to the exchange of the terms 'points' and 'lines' (along with the accompanying terms like collinear, coincidence, etc.) in the theorems. To illustrate this, consider a simple axiom of projective geometry: 'any two distinct points are incident with just one line'. When the terms 'lines' and 'points' in the theorem are interchanged, the following statement results: 'any two lines are incident with at least one point'. The resul-

tant statement can also be considered as one of the axioms for further elucidating and verifying the theorems and statements of projective geometry[9]. It can be observed that these two statements, which are considered as duals of each other, are completely different: one is about collinearity and the other on the property of coincidence. Other axioms of projective geometry can also be similarly dualized. Hence, it is not a surprise that for every theorem in projective geometry, a corresponding dual theorem exists. This duality of projective geometry holds good even for geometric figures. To provide a simple illustration, consider the figures quadrilateral and quadrangle. Quadrilateral is a system of four lines coinciding at six distinct points. In contrast, quadrangle is a system of four points that are joined by six lines[9]. Important aspects of description of a quadrangle (e.g. opposite sides and diagonal point) get inverted in the case of quadrilateral (opposite vertices and diagonal line). 'Inversion' here refers to the interchange of points and lines in the definition of important aspects (like 'diagonal' or 'opposite') of these 'dual configurations'[10].

As the above illustrations highlight, a statement (about an axiom, a theorem or a configuration) in projective geometry yields another statement that is still meaningful. This characteristic of the statements is captured by the principle of duality which states that 'every definition remains significant, and every theorem remains true, when we interchange the words *point* and *line* (and consequently also certain other pairs of words such as join and meet, collinear and concurrent, vertex and side and so forth)'[10]. This is the principle of duality in two-dimensional space. In three-dimensional projective

Table 1. Truth table of AND

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 2. Truth table of OR

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 3. Truth table of AND after swapping T and F

F	F	F
F	T	T
T	F	T
T	T	T

geometry, similar duality is present between lines and planes. This characteristic possibility – of swapping the terms – is observed in projective geometry alone because any two lines (even parallel) always have a unique incidence point. This unique feature of projective geometry fundamentally relates points and lines. Given that the dual of an axiom is still an axiom, historically, the dualizing process has been considered as a legitimate justification principle to validate theorems. In the 19th century Joseph Diez Gergonne, recognized this aspect about dual theorems and considered duality as a 'universal' principle[11].

Functional analysis: This is another domain where duality is invoked. Here, duality is observed between vector spaces and linear functionals described over them. In order to describe the duality, we will first briefly introduce the idea of vector spaces and linear functionals. Functional analysis, which is a prominent branch of abstract algebra, deals with operations prescribed over abstract spaces. An abstract space consists of a set of abstract elements ('whose nature is left unspecified') that possess some structure (like the distance function)[12]. What differentiates an abstract space from a set is the presence of some structure among its elements. This kind of general definition of space provides the flexibility of accommodating not only the obvious kinds of spaces – Euclidean, complex space – but also other sets of entities as 'spaces' like real line, complex plane, function space, etc. Similarly, a set of vectors constitutes a 'vector space' that has a structure. Here, the structure is provided by two algebraic operations, namely addition of vectors and multiplication of vectors by scalars[12]. Functionals are mapping operators that map an element of a vector space to a scalar entity. Here, the domain of functionals consists of vector spaces and the range is either a real line or a complex plane. Once vector spaces and functionals defined over them are given, duality found amidst abstract spaces can be described. As mentioned, in the abstract understanding of spaces, any set of entities having some structure can be considered as a space. In this way of thinking, it can be shown that the set of linear functionals defined over a vector space itself forms another vector space. If X is a vector space, then the functionals $f_1, f_2,\ldots$ defined over this space constitute a space since the two operations mentioned above – addition of entities and multiplication of entities by scalar values – can also be defined meaningfully for these functionals. Therefore, for a vector space X, there is an accompanying vector space X^1 constituted by all the functionals, each of which maps the elements of X to either a real line or a complex plane. This X^1 is called the algebraic dual space of X[12].

Group theory: In group theory, duality can be formulated amidst theorems which pertain to homomorphisms between groups. To illustrate the duality in group theory, consider an example provided by MacLane[13].

Theorem 1: The abelian group F is *free* if and only if, whenever $\rho\colon B \to A$ is a homomorphism of an abelian group B onto an abelian group A and $\alpha\colon F \to A$ a homomorphism of F into A, there exists a homomorphism $\beta\colon F \to B$ with $\rho\beta = \alpha$ (Figure 1).

Theorem 2: The abelian group D is *infinitely divisible* if and only if, whenever $\lambda\colon A \to B$ is an isomorphism of an abelian group A onto an abelian group B and $\alpha\colon A \to D$ a homomorphism of A into D, there exists a homomorphism $\beta\colon B \to D$ with $\beta\lambda = \alpha$ (Figure 2).

The above two theorems, MacLane indicates, can be considered as 'dual' to one another since one theorem under a transformation process results in the other. This transformation process includes two activities. First, the terms in a theorem (e.g. homomorphism like α) need to be swapped by other terms (e.g. isomorphism like λ). Second, direction of the functions (arrows in Figures 1 and 2) and order of the factors in the products have to be reversed[13]. Since both theorems transform to one another in the above sense, MacLane states that free abelian groups and infinitely divisible abelian groups are dual to one another. With this illustration, he proposes the general strategy to 'dualize' a theorem in group theory: for any statement about groups that refers only to homomorphisms, quotient groups and projection, a corresponding dual statement can be formed by interchanging the existing terms with isomorphisms, subgroups and injections respectively along with changing the order of the products involved[13]. MacLane demonstrates how this procedure can be used to formulate dual theorems in several other cases. For instance, a theorem about direct
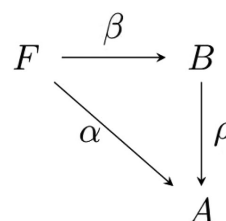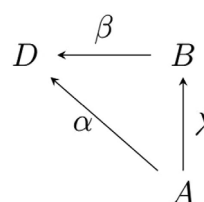

Figure 1. Theorem 1.

Figure 2. Theorem 2.

product of groups can be shown to be dual to a theorem about free product of groups. Similarly, theorems about ascending and descending central series of groups can be shown to be dual to one another. Since the duality in group theory is articulated amidst statements about specific group concepts and operations, theorems of group theory are the duals. In this sense, the duality found in group theory is similar to the one found in projective geometry.

Category theory: Even though category theory is a recent addition to abstract algebra, since the mathematical apparatus found in this theory allows for providing foundation for other algebraic theories, categories have become important mathematical objects. Therefore, it is necessary to determine how duality in this theory is articulated. A category is a mathematical object which consists of certain entities a, b, c, … and maps f, g, h …, which are also called as 'arrows', between these entities[14]. These arrows satisfy the following conditions:

- For every object a, there is an identity arrow such that $1_a : a \rightarrow a$.
- If $f : a \rightarrow b$ and $g : b \rightarrow c$, then a composite arrow can be defined such that $g*f = a \rightarrow c$.
- The composite arrows respect associativity such that $f*(g*h) = (f*g)*h$.

The notion of duality in category theory is articulated in the context of equivalence of categories. In order to define the notion of equivalence found here, few other basic concepts need to be introduced first. In category theory, homomorphisms among categories are called functors. Since categories consist of not only entities but also arrows between them, a functor between two categories $F : C \rightarrow D$ maps both the entities and the arrows of C to that of D. Given these functors F, G, …, a further notion of natural transformations, maps amidst these functors, can be defined[15]. Defining these maps allows for formulating functor category $[C, D]$ consisting of functors and the maps between them. In this category $[C, D]$, there are few maps that are isomorphisms and these are labelled as natural isomorphisms.

With the above brief introduction to some concepts, different notions of equality for categories can be defined. The basic notion of equality between two categories implies that the entities and the maps of both the categories are the same. Since the criterion is 'unreasonably strict', this notion of equality is not suitable for categories[15]. A more interesting and useful notion of equality can be defined using natural isomorphisms. Two categories C and D are isomorphic if the functors between them $F : C \rightarrow D$ and $G : D \rightarrow C$ are such that

$$F*G = 1_D \quad G*F = 1_C,$$

where 1_D and 1_C are identity functors of the respective categories[15]. However, $C \cong D$, where $\cong$ represents isomorphism is still 'too strict' for categories[15]. Hence, a much more relaxed notion called equivalence is defined. Two categories are equivalent, represented as $C \simeq D$, when the two functors F and G are related such that

$$F*G \cong 1_D \quad G*F \cong 1_C.$$

As can be seen, unlike the scenario of $C \cong D$, where the functors were equivalent to the identity functors, in $C \simeq D$ the functors are just isomorphic to the identity functors. This kind of equality returns the original thing only up to isomorphism, not 'on the nose'[15]. With equivalence amidst categories understood, duality in category theory can be defined. An equivalence of the form $C \simeq D^{op}$ is considered as duality between these two categories. Here, D^{op} represents an opposite category of D because it has all the entities of D but with arrows reversed. A concrete example of this duality is illustrated by the Stone duality theorem, which states that the category of finite boolean algebras is equivalent to the opposite of the category of finite sets.

Duality in physics

Among the disciplines in science, it is in physics where the concept of duality is put to use in numerous interesting ways. Among these, the prominent one is the wave–particle duality of radiation and matter formulated during the initial phase of quantum mechanics. Since this concept played a central role in the development of 20th century physics[16], duality was considered important and relevant for physics. Given the centrality of wave–particle duality, this version of duality has been extensively analysed, both historically and philosophically. As these studies show, 'wave–particle duality' is a wider label under which numerous different formulations – those of Einstein, Bohr, Heisenberg, etc. – are bracketed. Hence, there is no single particular definition of wave–particle duality and its historical evolution itself is an interesting case-study for how a scientific concept acquires multiple, yet related, meanings. In a separate study, we will examine the multiple meanings of wave–particle duality and how these different connotations are similar to the ones discussed here (https://www.current-science.ac.in/php/forthcoming/2020/2019000420.pdf).

Apart from wave–particle duality, there are other instances of duality found in physics. In certain areas of modern physics, especially in the domains of quantum field theory (QFT) and string theory, there are theories which are considered to be dual. This kind of duality consists of two theories that are related to each other in several ways. First, these two theories are about the same phenomenon and are theoretically equivalent. Here, 'theoretical equivalence' briefly means that there is no difference in the final description of the phenomenon or system provided by these two theories. However, being theoretically equivalent does not imply that they are the

same theory through and through. Even though they provide the same outcome, there might be differences in the way they achieve this and in the physical description of the phenomenon. Thus, these two theories might differ in the objects they use, their properties, the range within which they give appropriate values, etc. Amidst these differences, if they are 'dual' theories, there is a particular mapping that relates these differences between the two theories. It is the presence of this mapping that is the central characteristic of this duality.

The following example illustrates the above duality. Consider a system whose Hamiltonian can be articulated using two different fields $-\phi$ and ϕ^*. These fields describing the same system are related in a way as described below. The perturbation series of this system expressed in terms of these fields respectively will be

$$H = H_0 + gH_1 = H_0^* + g * H_1^*.$$

Here, g and g^* are the coupling factors and are related to each other in the following way: $g = 1/g^*$. Because of this reciprocal relationship between g and g^*, the representations of the system through the fields ϕ and ϕ^* will also have a reciprocal relation. This reciprocal relation between the two ways of talking about the system provides two alternative ways to study the same system. Given that both these approaches are equivalent descriptions of the system, when g is very large, the system can be analysed through g^*, given the perturbation series would be more accurate in this form[17]. This possibility – of having two theories/representations for the same phenomenon and these two theories being related through an 'equivalence map' such that the weak coupling regime of one theory is equivalent to the strong coupling regime of the other theory – is being referred to as duality here[18]. This kind of duality that is usually encountered in QFT and string theory is labelled as strong/weak duality or S-duality, where S stands for the symmetry group $SL(2, Z)$.

To illustrate one instance of S-duality, consider the duality between electric (E) and magnetic (B) fields. The similarity between E and B became evident with Maxwell's theory. Specifically, it was easy to observe through Maxwell's equations that both these fields can be transformed to one another under the following transformation: $E \rightarrow B$ and $B \rightarrow -E$ (ref. 18). This 'duality transformation' between E and B implies that the interpretations of a field either as 'magnetic' or 'electric' are equivalent descriptions. The equivalence suggested that there is one-to-one correspondence between the sources and charges of the fields. Because of this duality, the equivalent of electric charges (q) for magnetic fields, namely 'magnetic charges' (g), were proposed. In 1931, Dirac by demonstrating this equivalence within the quantum framework, provided the quantization condition for the quantum field which brought out clearly the relation between g and q as $qg = 2\pi n$, where $n = 0, \pm1, \pm2, \ldots$. As

shown by this relation, the charges are inversely related to one another. The complete equivalence between electric and magnetic fields in QFT was demonstrated in 1970s and was possible to achieve through other subsequent accomplishments. The equivalence between sine-Gordon theory and massive Thirring model, theories about two-dimensional fields, was shown by Coleman and Mandelstam in 1975. Specifically, it was shown that states of solitons in sine-Gordon theory were equivalent to the states of elementary particles described by the massive Thirring model. The coupling constants of the fields given by each of these theories exhibited the weak/strong duality relation, where the coupling constant of one theory in its weak regime corresponds to the coupling constant of the other theory in the strong regime[18]. This equivalence was further generalized for four-dimensional field theory in 1977 by Montonen and Olive, who showed evidence for the dual symmetry between two formulations of the same theory such that the magnetic and electric charges swap under this transformation. Apart from the duality of electric and magnetic fields, there are other interesting illustrations of dual theories in physics. For instance, string theory exhibits T-duality; Ads/CFT duality is present between QFT and string theory[18].

Duality in other branches of science

The concept of duality is invoked in other branches of science as well to capture specific characteristics of entities or processes. We will briefly mention some of them here for the sake of completeness. In biochemistry, certain molecules are said to exhibit 'functional duality' when they are key initiators of not one but two unique processes[19]. Similarly, 'mechanistic duality' is formulated in chemistry to capture the dual ways – either homolytically or heterolytically – in which a chemical reaction can proceed[20]. In ecology, the 'biotope space' (the physical space in which a species is found) and the 'niche space' (an abstract space defined uniquely for a species) are supposed to exhibit a reciprocal relation that is termed as 'Hutchinson's duality'[21]. In electrical engineering, electrical equations exhibit 'duality' under certain transformation rules. In an equation, when voltage terms are replaced by current terms, resistance with conductance, capacitance with inductance, etc. the resulting equation is equivalent to the original equation[22]. In mechanical engineering, structural mechanics formulates a similar kind of duality between stress and strain concepts. The equations derived through the kinematic analysis of rigid-body systems and the ones through static analysis exhibit static–kinematic duality[23].

Duality, symmetry and completeness

In the previous section, the various formulations of the concept of duality in different disciplines have been

discussed. The multiple meanings of duality across various domains in science should not be surprising. Scientific concepts often show this diversity of meanings. We could consider the simple example of mass and note its many meanings across different theories[24]. The meaning of mass prior to Newton is different from the ways it is used in his work. (In fact, there are multiple meanings of 'mass' already present in Newton's laws of motion: inertial mass, mass as related to the action of force, passive and active gravitational mass.) There are other meanings of 'mass' found in electrodynamics, relativity theory and QFT. Similar to how all these different meanings are coalesced under one concept called 'mass', so is the case with 'duality'. Why is the same term 'duality' used across theories even though it is evident that the meanings attributed to it are different in different cases? One way to interpret this usage is to consider it as a strategy of minimizing technical terms, a kind of parsimony in the use of concepts within science. According to this interpretation, the concept of 'duality' is made up of a bundle of characteristics and each of the instances discussed above qualifies as an instance of 'duality', since some of these characteristics, even though not all, would be features of the instance. In other words, the several connotations do possess certain family resemblances that make it possible to bring them together under the same concept[25]. The other possibility could be that there is really something intrinsic to all the scenarios that make scientists refer to them, and not to others, as 'duality'. Next, we will argue that both the above-mentioned motivations are found in the usage of duality across disciplines.

In order to bring out these aspects, we will first analyse the nature of relations present in the various scenarios discussed in the previous section, and identify three different kinds of duality relations. The first two types of duality relations are characterized by similarity and inversion relations respectively. The third type of duality relation possesses both the characteristics of being similar and being inverse. Subsequently, we will interpret the last kind of duality relation through the notion of completeness.

All the different types of dualities discussed above share the same structure which can be represented schematically as $D_1 \leftrightarrow D_2$, indicating the duals – D_1 and D_2 – that are connected through the relation of duality. Since none of the dualities found in science exhibits the trivial cardinal duality, '$\leftrightarrow$' has a unique meaning depending on the context. Thus, analysing duality requires understanding this relationship between the duals. It can be noticed that none of the instances of duality discussed in here instantiates strict equality between their duals. Instead, in some instances (like the ones observed in classical logic, projective geometry and dual theories of physics), closer cognates of equality – equivalence or similarity – are found. Also, not surprisingly, the other extreme form of relation – the duals being contrary of one another – is also not observed in the spectrum of connota-

tions discussed here. The relation between the contrary duals, like good–bad, hot–cold, dead–alive, etc. is often understood through the concept of negation. According to this interpretation, the pairs are immediate contraries and the negation operator transforms one to the other[26]. (The qualification 'immediate' here is important because some contraries, like weak–strong and tall–short, can have intermediary values between the two extremes.) It is evident that none of the versions of dualities considered here can be understood as a pair of immediate contraries related through the operation of negation. Among the instances discussed above, dualities found in classical logic, projective geometry, group theory and category theory, and dual theories of physics do involve a kind of inversion or reversal. However, the inversion relation involved in these cases is quite unlike the strict form of negation. Therefore, the non-obvious instances of duality discussed here do not seem to take either of the extreme connotations of 'being equal' or 'being opposite'. Instead, duals are characterized by the weaker variants of these extremities: either the duals are similar to one another, or are antithetical in a peculiar way. Apart from these two kinds, there are some duality relations which have both the characteristics. That is, neither of the characteristics – similarity or inversion – individually captures the essence of the duality relations found in logic, projective geometry and dual theories in physics. The duals in these cases, so to speak, occupy a *liminal region* between the extremes of equality and opposition, such that they are not completely different from one another but at the same time are related through some sort of inversion. In this kind of duality, it appears that the transformation plays an important role since it brings about the dual characteristics of the relation: the duals that are similar to one another become inverse (in a specific sense) after the transformation.

When the liminal duality relation is articulated in the above manner, it seems to share traits with symmetry, since here too transformations play an essential role. In fact, among the several definitions and instances of symmetry, there is a historically interesting one that seems particularly relevant for showing the similarity between duality and symmetry. Usually, symmetry is defined as invariance under a transformation[27]. However, historically, symmetry possessed several different meanings. Euclid used it to represent proportionality; for Vitruvius, symmetry was the correspondence among the parts of a whole; in the 17th century, it largely stood for the equality of the parts (bilateral symmetry)[28]. In all these various usages, the core meaning of symmetry, as a property of the whole, was intact. In the 18th century, symmetry also acquired the connotation of equivalence among independent entities. This historically interesting development in the meaning of symmetry is contextualized in the difficulties regarding the notion of equality found in solid geometry. The accepted definition of equality for solid

figures (three-dimensional polyhedra) was given by Euclid. According to this definition, two solid polyhedra, like tetrahedrons, are equal if they are 'contained by similar planes equal in multitude and in magnitude'[28]. However, it was realized that this principle of equality is not sufficient since equality of the planes that constitute the polyhedra does not guarantee the superposability of these two figures. This problem arises because it is not only the equality of planes, but also the order of their arrangement, while constituting the polyhedron, that plays an important role. Interestingly, the same problem was also confronted by Immanuel Kant, but in a completely different context. In order to support Newton's conception of space, Kant proposed that physical space can be characterized by a unique property called directionality[28]. Specifically, he showed that the right-hand and left-hand images in a plane are incongruent to one another, even though they are mirror images. Coming back to the question about the equality of polyhedra, in the 18th century, Adrien-Marie Legendre attempted to resolve this problem by proposing a new notion of equality. For him, two polyhedra are equal only if they are equal in magnitude and are congruent (i.e. superposable). In contrast, the figures which are not superposable, Legendre argued, are still equal, but the principle at work is equality by symmetry[28]. By proposing this novel definition of equality, Legendre brought a change in the notion of symmetry. Apart from the invariance of the whole, symmetry also acquired the meaning of a specific kind of relation between two distinct entities.

This brief account provides yet another illustration of how concepts acquire new meanings. Nevertheless, the main intention of providing this historical development is to highlight the similarity between duality and symmetry. The problem faced by Legendre, or equivalently, the one articulated by Kant are strikingly similar to the duality scenarios encountered in logic, projective geometry and dual theories of physics. The two 'symmetrical' polyhedra are constituted by the same set of planes. The only difference between these two is in the order in which the planes are arranged to constitute the solid angles. For a particular order found in one polyhedron, the other one is supposed to have the 'inverse' order[28]. Therefore, this pair of polyhedral – and also the planar counterparts, the left-hand and the right-hand images – can be considered as archetype instances of dual objects: the duals are not equal to one another (since they cannot be superposed on each other), nor are they completely different; they are similar (since they are constituted by the same elements), but are 'opposite' to one another (with regard to the order of the planar arrangement).

Does this imply that the notion of symmetry sufficiently explains duality? We do not think so. The similarity between duality and symmetry was grounded on the importance of transformations for both these concepts. Regarding symmetry, two entities are symmetrical if there is a transformation that relates them through some invariance. So what is the role of transformations in the context of duality? When the examples of duality are carefully observed, the related transformations only show or demonstrate how the duals are related in a particular way. There is also another important argument for not reducing duality to the transformation. The interpretation of duality through its transformation projects it as a relation that acquires two different characteristics before and after the transformation. This is an incorrect representation, since the liminal duality relation does not switch its nature in this fashion. Instead, the relation possesses these characteristics simultaneously. This aspect of duals – to be simultaneously similar and inverse to one another – is an essential feature of duality and to provide its meaning is to explicate the nature of this feature.

One way to understand this unique feature is to consider the notion of completeness. What is clear from the examples and our analysis is that the idea of duality seems to 'naturally' arise in cases where the duals together offer a complete description. We argue that two entities that constitute duality have the following features: (1) the duals complete each other, and (2) the duals together exhaust the possibilities. Completeness is also a concept that has many uses in mathematics and the sciences. The principle of completeness is important in logic, real analysis, topology, computing, quantum theory and so on. While all these notions of completeness do not entail duality, in duality, there is nevertheless a sense of completeness. When the various instances of duality discussed here are considered, it is possible to distinguish two different notions of completeness: (1) ontological completeness, where the duals exhaust the possible types of entities and ways of existence (as observed in the case of mind–body duality or atman–brahman duality), and (2) epistemological completeness, where there are only two irreducible descriptions of the world (similar to that of string theory or QFT). To illustrate the aspect of mutual exhaustion observed between duals, consider the example of duality observed between right-hand and left-hand planar images. As Kant argued, every figure in a two-dimensional plane has the property of directionality (also known as 'chirality') and this property can have only two values, which can be phrased for convenience as 'downward' and 'upward' directionality. This implies that an image can be placed on a plane in two ways, either facing downward or upward. The right-hand and left-hand images, thus, differ from one another only with respect to this property. Given this, it can be mentioned that the right-hand and left-hand images constitute a duality, since they exhaust the dual possibilities of a particular characteristic (directionality). This aspect of duality can be phrased in the context of other examples too. Thus, through these examples and analysis, we show the rich semantic plurality that is an important part of conceptualization in the sciences.

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