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Dynamics of bow-tie shaped bursting: Forced pendulum with dynamic feedback

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A detailed study is performed on the parameter space of the mechanical system of a driven pendulum with damping and constant torque under feedback control. We report an interesting bow-tie shaped bursting oscillatory behaviour, which is exhibited for small driving frequencies, in a certain parameter regime, which has not been reported earlier in this forced system with dynamic feedback. We show that the bursting oscillations are caused because of a transition of the quiescent state to the spiking state by a saddle-focus bifurcation, and because of another saddle-focus bifurcation, which leads to cessation of spiking, bringing the system back to the quiescent state. The resting period between two successive bursts (T_{rest}) is estimated analytically. *Published by AIP Publishing.*
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The simple pendulum with damping and torque is an instructive system to study because it exhibits rich dynamical behaviour. It is interesting not only because of its equivalence with the resistive capacitative shunted junction (RCSJ) model of a single Josephson junction (whose diverse applications range from its use in SQUID magnetometers to quantum logic circuits) but also because it finds applications in robotics, providing insights for control mechanisms. Here, we report bow-tie shaped bursting oscillations, which are produced when the damped pendulum with a torque receives an external periodic forcing whose phase is instantaneously shifted through a dynamical feedback. In an earlier work,[7] we had reported these bow-tie shaped bursts in a system of 2 resistively coupled Josephson junctions. Our finding in the present work of the same bursts in the simpler or reduced system of the pendulum gives a better and clearer insight into this bursting mechanism arising from the modulation of the fast oscillations of the pendulum by the slow driving frequency regulated by a dynamic feedback.

I. INTRODUCTION

The damped simple pendulum under external forcing has been studied extensively over the years (see, for example, Ref. 1 and references therein). It continues, however, to evoke immense interest because of the numerous unexplored modes of oscillation and dynamical behaviour it can exhibit.[1–3] Under torque, the system displays further richness in its complex behaviour and acts as the mechanical analogue of the resistive capacitative shunted junction (RCSJ) model[4,5] of the Josephson junction. The pendulum under different conditions has been used to represent different nonlinear processes and has been used as simplified models of biological systems.

In this work, we investigate an interesting bow-tie shaped bursting oscillation, which is produced in a certain parameter regime of a forced pendulum with linear velocity damping, under constant torque and dynamic feedback. Similar bursts of identical bow-tie shape have been reported by us previously[6,7] in a system of *two* resistively coupled Josephson junctions in a certain parameter regime when one junction is kept in an oscillatory mode and the other in an excitable (shunted) mode and in a network of Josephson junctions.[8] In these latter systems, however, the bursts were intrinsic, although the oscillatory junction acts as a driver for the excitable junction and also receives the latter's output through its coupling; there was, however, no external forcing.

Bursting oscillations have been extensively studied and classified (see, for example, Refs. 9 and 10). In neurons, it is believed that bursts play an extremely important role in neuronal communication, in motor pattern generation, for encoding different features of sensory input, etc.[10–12] Yet, not all electrophysiologically possible bursts have been classified nor are all the possible mechanisms producing the bursts exhaustively known.

The investigations reported in this paper were motivated by our attempt to find the bow-tie shaped bursting oscillations reported in Ref. 7 in a reduced or simpler system.

We believe that our results are useful and of significance as we suggest a possible bifurcation mechanism to understand a *forced* bursting phenomenon having the intriguing bow-tie shape, which has not been discussed before in the literature available thus far in the context of a forced system.

The system we study is described by the following second order differential equation for the externally forced damped pendulum with torque:

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$$\ddot{\theta} = I - \sin(\theta) - \alpha\dot{\theta} + \sin(\omega t - \theta), \quad (1)$$

where I denotes the constant torque, α is the damping parameter (both I, $\alpha > 0$), the dot denotes the derivative with respect to time, and ω denotes the forcing frequency.

The system is modulated by a dynamic feedback, which enters via the external forcing term through the argument of the sine function $\sin(\omega t - \theta)$.

The equation is then equivalent to the RCSJ model of a single Josephson junction driven by a sinusoidal forcing term whose phase is dynamically shifted by the phase difference θ between the macroscopic wave functions on either side of the junction. To the best of our knowledge, bursting behaviour of the kind we report here is not known yet for this system in the available literature.

We investigate here the dynamical bifurcation mechanism producing the bow-tie shaped bursts in the *forced* system, and we conduct an extensive study of its parameter space.

II. THE FORCED SYSTEM WITH DYNAMIC FEEDBACK

Introducing $\tau = \omega t$, Eq. (1) may be rewritten as three first order autonomous differential equations

$$\dot{\theta} = V$$
$$\dot{V} = I - \alpha V - \sin\theta + \sin(\tau - \theta)$$
$$\dot{\tau} = \omega. \quad (2)$$

The equations are solved numerically and V is plotted in Fig. 1 for a specific set of parameters. We observe that V displays bow-tie shaped bursting oscillations in a certain range of the external forcing frequency. It may be noted that to obtain the bursts, the driving frequency has to be restricted to values that are much smaller than the oscillator's natural frequency. The bursting and the resting phases increase with decreasing ω.

The multimedia files linked to Fig. 1 (and Fig. 5 later on) illustrate the bursting behaviour with changing ω and I, respectively. In Secs. III–V we investigate the dynamics underlying these intriguing bursts.

The system in Eq. (2) has two fixed points (θ^*, v^*, τ^*), which exist only for $\omega = 0$, namely, $(\arcsin(I/2), 0, 0)$ and $(\pi - \arcsin(I/2), 0, 0)$. Furthermore, they exist for $I \leq 2$, with the system having no fixed points for $I > 2$. At the first fixed point P_1: $(\arcsin(I/2), 0, 0)$, the eigenvalues λ of the linear stability matrix are found from the characteristic equation,

$$\lambda(\lambda^2 + \alpha\lambda + 2h) = 0, \quad (3)$$

where $h = \sqrt{1 - (I/2)^2}$. Therefore, one of the eigenvalues is zero. The other two eigenvalues are

$$\lambda_{+,-} = (-\alpha \pm \sqrt{\alpha^2 - 8h})/2. \quad (4)$$

The first term, $-\alpha/2$, is always negative for both $\lambda_{+,-}$ since $\alpha > 0$. For the second term, it can be seen that when $\alpha < 2\sqrt{2h}$, the two eigenvalues form complex conjugate

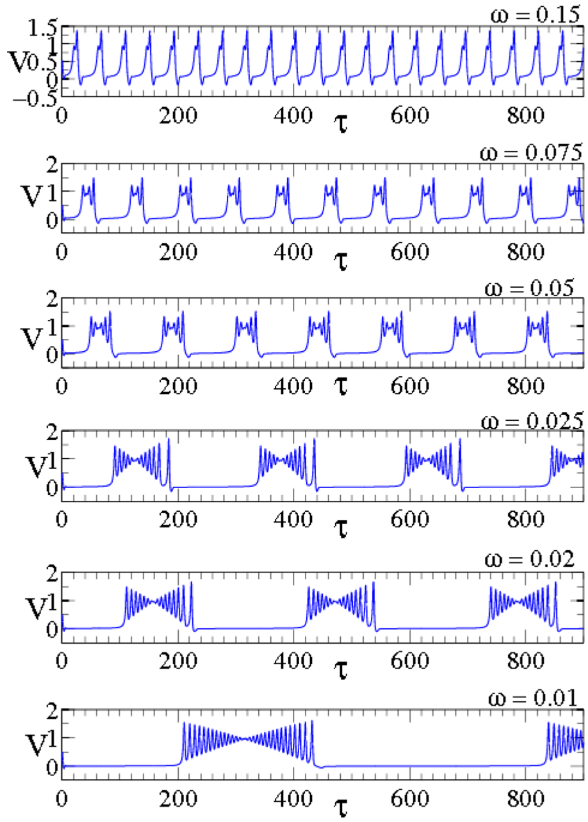

FIG. 1. Time series of the variable V in Eq. (1) for different values of the parameter ω, the frequency of the external drive. The values of $I = 1.15$ and $\alpha = 1.2$ in all the plots. (Multimedia view) [URL: http://dx.doi.org/10.1063/1.4971411.1]

pairs and the fixed point is a stable focus. The critically damped limit is given by $\alpha = 2\sqrt{2h}$. For $\alpha > 2\sqrt{2h}$, the fixed point is a stable node. This is analogous to the over-damped limit of the damped driven pendulum.

For the other fixed point $P_2 : (\pi - \arcsin(I/2), 0, 0)$, $\cos(\theta^*) = -\sqrt{1 - (I/2)^2} = -h$, and the characteristic polynomial is $\lambda(\lambda^2 + \alpha\lambda - 2h) = 0$. Here too one eigenvalue is zero. The other two are

$$\lambda_{+,-} = (-\alpha \pm \sqrt{\alpha^2 + 8h})/2. \quad (5)$$

The second term will always be real for $I \leq 2$. Therefore, λ_- is always stable. For λ_+, the eigenvalue is negative only when $\alpha > \sqrt{\alpha^2 + 8h}$ that is clearly not possible and α will always be less than $\sqrt{\alpha^2 + 8h}$ for all $I \leq 2$. Hence, P_2 is a saddle point.

III. BIFURCATION MECHANISM PRODUCING THE BURSTS

The bursts occur for a certain range of $\omega \neq 0$. As the system has no fixed points for $\omega \neq 0$, one could instead look at the $\theta - V$ phase space at different instants of time. At any given instant of time, the vector field will look similar to that of a damped pendulum under constant torque. This is shown in Fig. 2 where the vector field on the $\theta - V$ plane for different increasing times t is illustrated. The red curve is the V nullcline at that instant of time and the green curve is the

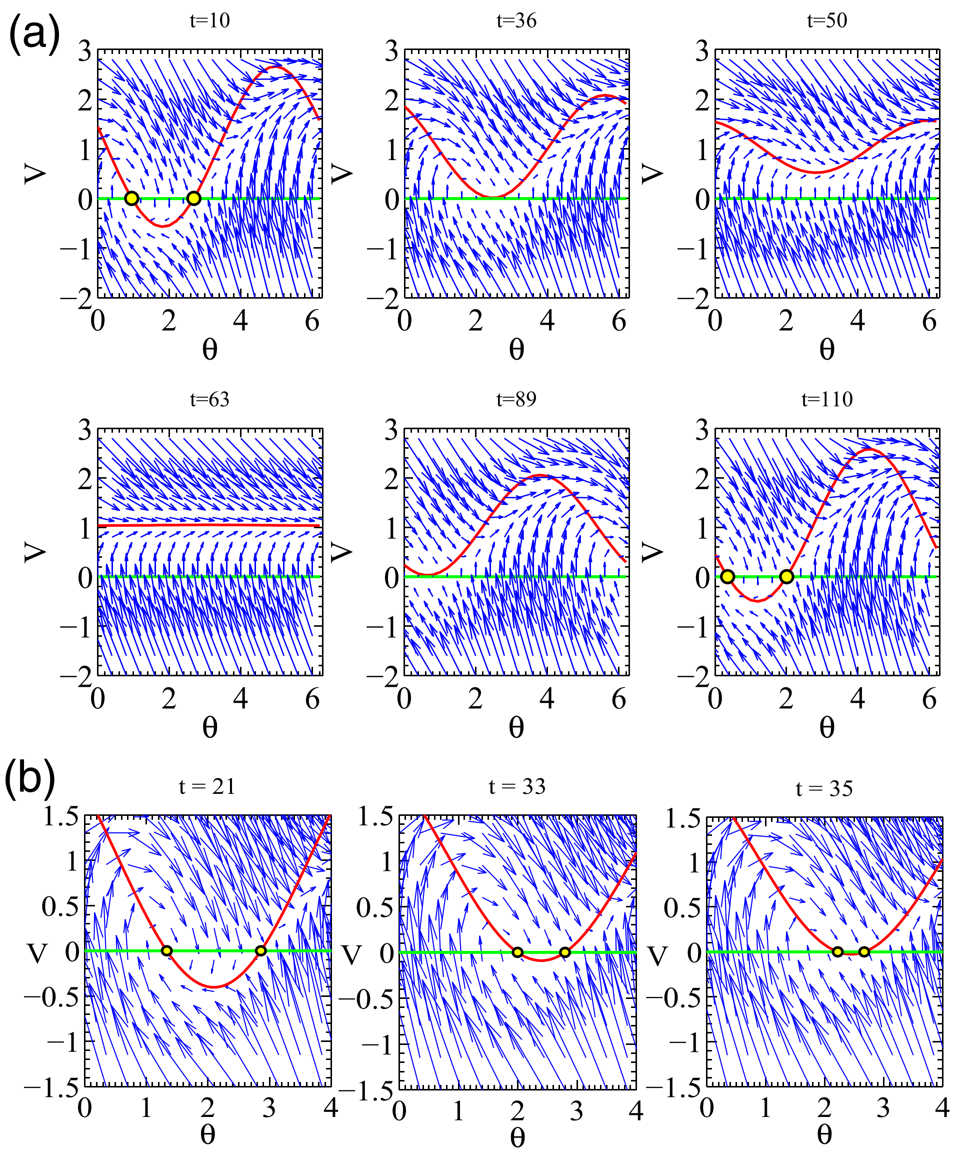

FIG. 2. The vector field of the system in the $\theta - V$ plane at different times t showing evolution with time ($I = 1.25$, $\omega = 0.05$). V-nullcline is in red; θ-nullcline is in green. Intersections between the nullclines are points of constants θ and V. (a) Creation of the fixed points P_1 and P_2 via a saddle-focus bifurcation, and their subsequent merger and disappearance via another saddle-focus bifurcation leading to oscillatory solutions again. (b) Close-up of (a) showing the detail of saddle-focus bifurcation. (Multimedia view) [URL: http://dx.doi.org/10.1063/1.4971411.2] [URL: http://dx.doi.org/10.1063/1.4971411.3]

θ nullcline, obtained by setting $\dot{\theta}$ and $\dot{V}$ on the left hand side of Eq. (2), respectively, to 0. This yields for the θ curve: $\dot{\theta} = V = 0$, and for the V nullcline V_n

$$V_n = \frac{1}{\alpha}\big(I - \sin(\theta) + \sin(\omega t - \theta)\big). \tag{6}$$

The intersection of these two curves represents a point where both θ and V are constants at that instant of time. This point can be looked at as an "equilibrium point" of the system on the $\theta - V$ plane. Clearly, there are two such points on the cylindrical manifold of the system. These two points P_1, P_2 are indicated by yellow dots in Fig. 2

$$P_1 := V^* = 0; \quad \theta^* = \sin^{-1}\left[\frac{I}{2}\sec\left(\frac{\omega t}{2}\right)\right] + \left(\frac{\omega t}{2}\right)$$

$$P_2 := V^* = 0; \quad \theta^* = \pi - \sin^{-1}\left[\frac{I}{2}\sec\left(\frac{\omega t}{2}\right)\right] + \left(\frac{\omega t}{2}\right). \tag{7}$$

P_1 behaves like a stable focus and P_2 as a saddle point in the $\theta - V$ plane. These two points approach each other and coalesce and disappear periodically leaving behind a smooth field. This happens due to the decrease in the amplitude of V-nullcline (red curve in Fig. 2) such that at a certain time (shown at $t = 36$ in the figure for $\omega = 0.05$), the V-curve and the θ-curve are tangent and beyond which they no longer intersect and the points P_1 and P_2 vanish. With increasing time, the amplitude of the V-curve continues to decrease until it reaches zero and becomes a straight line (around $t = 63$ in the figure) parallel to the θ-axis, which also corresponds to the θ-curve. Subsequently, the amplitude increases again until (after around $t = 89$) the two curves intersect again leading to the re-appearance of points P_1 and P_2.

For $\omega = 0$, the equations of motion reduce to that of a damped pendulum under constant torque. The merging of P_1 and P_2 and their consequent disappearance then occurs via a saddle-node bifurcation on an invariant circle. However, here there is an external sinusoidal forcing signal effected by the t itself which is driving the points P_1 and P_2 to merge and disappear and then to again reappear. From Fig. 2 (and from the movies linked to Figs. 2(a) and 2(b)), we clearly see that the vanishing of P_1 and P_2 and their subsequent re-emergence occurs via a saddle-focus bifurcation.

IV. RESTING PERIOD BETWEEN BURSTS

In the interval of time t when the V and θ nullclines intersect, all the trajectories of the system on the $\theta - V$ plane are attracted to the stable point P_1 and the oscillations are killed. Once P_1 and P_2 mutually collide and vanish, the solution becomes oscillatory. During this time span, the system exhibits oscillation about a mean value of V, which depends strongly on I and weakly to α. For a given I and α, the time span during which P_1 and P_2 exist, corresponding to the resting phase of the bursting oscillation in Fig. 1, can be estimated.

The V nullcline in Eq. (6) is a sinusoidal function in both the θ and the time variables. At a given time, it takes the form of a sinusoidal waveform about the $V = I/\alpha$ line in the $\theta - V$ space. At a certain point of time, when the amplitude of the V-curve is equal to I/α as measured from the $V = I/\alpha$ line, the nullcline is tangent to the x-axis (V = 0). This tangent point is a "fixed point" in the $\theta - V$ space, and every trajectory converges to this point. All oscillations are killed and the system goes into the resting phase. As time evolves further, the amplitude of the nullclines becomes greater than I/α and it intersects the x-axis at two distinct points (P_1 and P_2). These two points are the stable focus and the saddle point, and every trajectory continues to sink into the focus and no oscillations occur. After a certain point of time, the nullcline peaks with a maximal magnitude of $2/\alpha$ as measured from the $V = I/\alpha$ line. Beyond this time, its amplitude decreases until again its magnitude is I/α and the nullcline again becomes tangent to the x-axis when the focus and the saddle have coalesced to a single point. After this, the amplitude of the nullcline becomes smaller than I/α and no longer intersects with the x-axis. The system is then in the oscillatory phase of the burst. This cycle repeats itself periodically manifesting in the solution of the ordinary differential equation as periodic bursting and resting phases. The resting period of the burst can be estimated by considering the peak value V_p of the V nullcline in the $V - \theta$ plot in Fig. 2. This extremal value V_p of V_n with respect to θ may be found from Eq. (6). This gives

$$\frac{dV_n}{d\theta} = \frac{1}{\alpha}\left[-2\cos\left(\frac{\omega t}{2}\right)\cos\left(\frac{\omega t}{2} - \theta\right)\right] = 0$$
$$\text{i.e.,} \quad \theta = \frac{\omega t}{2} - \frac{\pi}{2}. \tag{8}$$

Substituting this value of the θ in the expression for the V-nullcline, its extremal value V_p is obtained as

$$V_p = \frac{1}{\alpha}\left[I + 2\cos\left(\frac{\omega t}{2}\right)\right]. \tag{9}$$

The graph of the evolution of the amplitude of the nullcline at a given point in space is shown in Fig. 3. When the curve first intersects the x-axis at the $V_p = 0$ line, it corresponds to the V-nullcline tangent to the x-axis resulting in the emergence of the focus and saddle fixed points in the $V - \theta$ space. This is the point when all oscillations are killed and the system is driven to resting phase. Let t_1 be the time when the function first intersects the x-axis. As the function continues to evolve, it spends some time in the negative half of the y-axis, and at certain point of time t_2, the function lifts itself

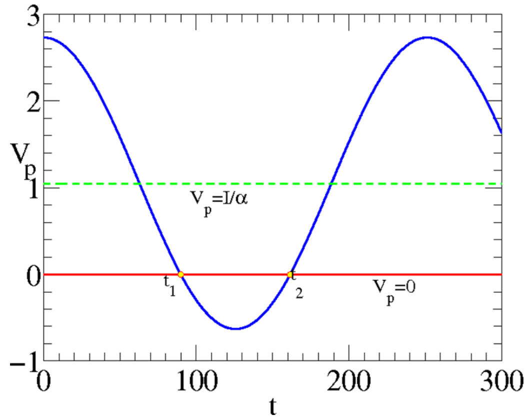

FIG. 3. Temporal evolution of the peak of the V-nullcline ($I = 1.25$, $\omega = 0.05$).

upwards, crosses the x-axis again, and becomes positive. This corresponds to the point on the $V - \theta$ plane when the V-nullcline is again tangent to the x-axis and the two fixed points collide and vanish and the system begins to oscillate again. At the points t_1 and t_2, the value of the function is zero. Therefore, the two points of time are simply given by

$$I + 2\cos\left(\frac{\omega t_1}{2}\right) = 0, \quad t_1 = \frac{2}{\omega}\cos^{-1}\left(\frac{-I}{2}\right) \quad \text{and} \tag{10}$$

$$I + 2\cos\left(2\pi - \frac{\omega t_2}{2}\right) = 0, \quad t_2 = \frac{4\pi}{\omega} - \frac{2}{\omega}\cos^{-1}\left(\frac{-I}{2}\right). \tag{11}$$

The resting period T_{rest} of the system is simply the time span between t_1 and t_2

$$T_{rest} = t_2 - t_1 = \frac{4}{\omega}\left[\pi - \cos^{-1}\left(\frac{-I}{2}\right)\right] \tag{12}$$

$$\text{or} \quad T_{rest} = \frac{4}{\omega}\cos^{-1}\left(\frac{I}{2}\right). \tag{13}$$

We notice that the resting period is independent of α, which holds throughout the solution regime, showing this characteristic bursting phenomenon. Fig. 4 shows the plot of the resting period against I for different ω. As expected, the resting period is maximum for low I and ω.

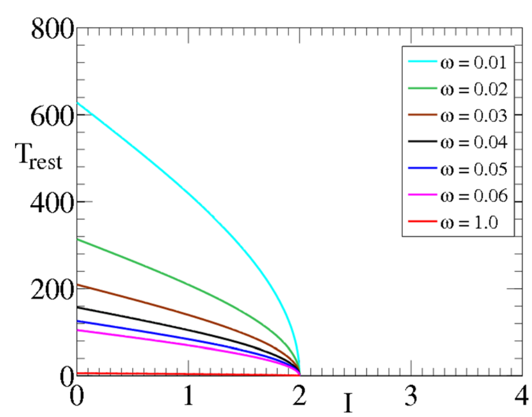

FIG. 4. Resting period of oscillations vs. I for various ω.

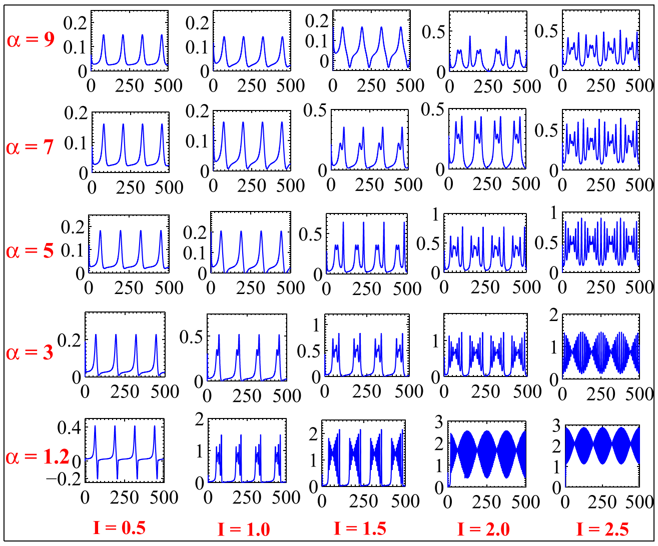

FIG. 5. The time series of the variable V for different sets of values of parameters I and α for $\omega = 0.05$. The time series for $\alpha = 1.19$, $\omega = 0.05$ with varying I is shown in the multimedia file. (Multimedia view) [URL: http://dx.doi.org/10.1063/1.4971411.4]

V. PARAMETER SPACE OF THE SOLUTIONS

The solution of the system exhibiting the cycle of bursting and resting is not the only possible solution. Indeed, this characteristic bursting solution is bounded in the parameter space. In different parameter regimes, one expects a different behaviour of the system. This is comprehensively captured in Fig. 5, which gives a snapshot of the entire possible range of the solutions in the $I - \alpha$ parameter space for a given value of ω. Clearly, apart from the characteristic bursting oscillations being discussed, there are other solutions where the resting period of the oscillation completely vanishes and the solution behaves like beats. Similar types of solutions have been observed for coupled Josephson junctions also. Furthermore, for extremely low values of I, the system exhibits single periodic spikes just as are observed in coupled Josephson junctions.[7] The actual boundaries of the different solutions in the $I - \alpha$ plane are shown in Fig. 6 for different values of ω. The solution regimes do not significantly change for different choices of ω. To understand the different regimes, we consider Eq. (6), which can be rewritten as

$$V_n = \frac{1}{\alpha}\left[I + 2\cos\left(\frac{\omega t}{2}\right)\sin\left(\frac{\omega t - 2\theta}{2}\right)\right]. \tag{14}$$

From this expression, it is very clear that the θ curve will intersect with the V curve ($V = 0$) only for values of $I \leq 2$ for all time t. Recall from our discussion earlier that bursting oscillations are killed only due to the existence of the points formed by the intersection of the two curves. For $I > 2$, the two curves do not intersect for any $\alpha > 0$ for all time $t > 0$. The system will therefore continue to exhibit oscillatory behaviour corresponding to the oscillating (spiking) part of the burst. The boundaries look remarkably similar to the corresponding parameter space plots of the unforced damped pendulum under constant torque.[2] There, the critical value of I, $I_c = 1$ demarcates the regime of stable solutions, showing periodic rotations from the stationary and bistable regions. In our forced system, the limit of I beyond which no bursting solution is possible is at the critical value $I_c = 2$, as can also be seen from Eq. (13) for the resting period between bursts. For large values of the damping, the boundary separating the bursting from the beating oscillations is not very sharp and clear-cut. This can be seen from the plots of Figs. 5 and 6.

Another very interesting feature is that for very small I at large α, there is a narrow strip of region in the $I - \alpha$ plane bounded by the green curve in Fig. 6 for which the solution of the system is closed; i.e., the system solution is characterised by closed orbit periodic oscillations, generally referred to as *librations*. This regime is unique for this system and has no correspondence with the generic damped pendulum under constant torque or the coupled Josephson junctions. This is due to the fact that the system is externally driven continually by a periodic forcing of angular frequency ω.

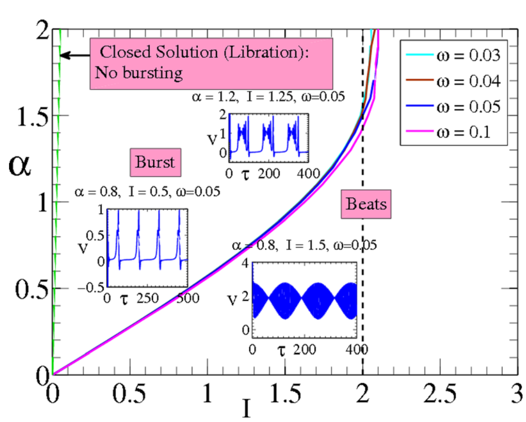

FIG. 6. The boundaries of the different solution regimes in the $I - \alpha$ parameter space for different values of ω.

VI. CONCLUSIONS

It is shown that the mechanical system of a forced pendulum with damping and constant torque exhibits bow-tie shaped bursting oscillations in the presence of a dynamic feedback. The bursts occur when the forcing frequency is extremely small in comparison to the natural frequency of the unforced system. We demonstrate that the bursts are formed via a saddle-focus bifurcation and they also terminate via the same mechanism. The resting period between two successive bursts (T_{rest}) is estimated analytically.

An exhaustive investigation of the parameter space of the system is performed, and the boundaries demarcating the bursting solutions from others are obtained. Different oscillatory behaviour emerges in different parameter domains.

The system studied also represents the RCSJ model of a single Josephson junction subjected to external periodic forcing, *and having its phase shifted dynamically by the difference in phase between the macroscopic wave functions on either side of the junction*. Although bursting oscillations are well known in different nonlinear systems including in neurons and in the Josephson junction and its arrays, the bow-tie shaped burst we have reported here occurring *in the presence* *of external forcing and feedback* is new. This system thus is ideal for reaching an understanding of the mechanisms underlying similar bursting oscillations in other complex systems (like Josephson junctions), which are of practical importance. Our results could also help in understanding neuronal bursting mechanisms in different situations incorporating dynamic feedback when one seeks to control the operating regime of the system (where each neuron receives inputs from an external source and from other neurons).

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